Critical Langevin dynamics of the $O(N)$-Ginzburg-Landau model with correlated noise

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Abstract. We use the perturbative renormalization group to study classical stochastic processes with memory. We focus on the generalized Langevin dynamics of the ϕ^4 Ginzburg-Landau model with additive noise, the correlations of which are local in space but decay as a power-law with exponent α in time. These correlations are assumed to be due to the coupling to an equilibrium thermal bath. We study both the equilibrium dynamics at the critical point and quenches towards it, deriving the corresponding scaling forms and the associated *equilibrium* and *non-equilibrium* critical exponents η, ν, z and θ. We show that, while the first two retain their equilibrium values independently of α, the non-Markovian character of the dynamics affects z and θ for $\alpha < \alpha_c(D, N)$ where D is the spatial dimensionality, N the number of components of the order parameter, and $\alpha_c(x, y)$ a function which we determine at second order in $4 - D$. We analyze the dependence of the asymptotic fluctuation-dissipation ratio on various parameters, including α. We discuss the implications of our results for several physical situations.

1. Introduction

For more than 30 years critical dynamics have been explored with field theoretical methods [1, 2, 3, 4, 5, 6, 7, 8, 9]. A variety of dynamic models were introduced to describe the collective evolution of systems close to critical points. Among these, the most common ones are the dynamics of non-conserved or conserved order parameters which successfully describe the evolution of uniaxial magnetic systems close to the Curie point or the dynamics of binary alloys close to the demixing transition, respectively. These problems as well as many of their generalizations discussed in [4, 5, 7, 9] are *classical* in the sense that their stochastic nature can be essentially ascribed to thermal fluctuations. Therefore, as in the simpler diffusion processes, the evolution of the interacting degrees of freedom — described via a field ϕ — is modeled by a functional Langevin equation in which the coupling to the environment is responsible for both thermal fluctuations, encoded in a stochastic external noise, and the friction. If the environment, which acts as a thermal bath for ϕ, is in equilibrium at a certain temperature β^{-1}, the space and time dependence of the friction coefficient and of the correlations of the thermal fluctuations are related via the fluctuation-dissipation theorem. The selected model bath determines then the remaining functional form of the noise-noise correlation and the usual choice is to take it to be delta-correlated in time which corresponds to white noise.

In cases in which the initial conditions of the system are drawn from an equilibrium Gibbs-Boltzmann distribution, or the system is allowed to evolve for a sufficiently long time such that this distribution function is reached, the space-time behavior of dynamic quantities is characterized by scaling laws in which the usual static exponents (ν, η etc.) appear but a new critical exponent z is needed to relate the space and time dependencies. Besides the analysis of equilibrium dynamics, field-theoretical methods allow one to study the *non-equilibrium* dynamics after a sudden quench from a suitably chosen initial condition to the critical point [10]. A Gaussian distribution of the initial field configuration with zero average and short-range correlations mimics a quench form the disordered phase [10]. A distribution with non-zero average but still short-range correlations describes (in the case of a scalar field ϕ) a quench from the ordered state [11, 12]. In these non-equilibrium cases a new critical exponent — usually denoted by θ and called the "initial slip exponent" — characterizes the short-time behavior of the average order parameter as well as of the correlation and response functions [10] (see, e.g., [13, 14, 15] for summaries).

In all the studies mentioned above, the noise is assumed to have a Gaussian distribution (as a consequence of the central limit theorem) with no temporal correlations, i.e., to be Gaussian and *white*. However, the coupling to thermal reservoirs yields, in general, non-Markovian Langevin equations in which the noise is correlated in time and the friction coefficient has some memory [16, 17, 18]. In many situations of practical interest a full treatment of such a *colored* noise is therefore necessary. For example, the escape rate of particles confined within a potential well crucially depends on the statistics of the thermal bath [19, 20, 21] as observed in the desorption of molecules

from a substrate undergoing a second-order phase transition which effectively provides a colored noise for the stochastic dynamics of the molecules (this phenomenon is called Hedvall effect and it was studied theoretically in, e.g., [22]). Another important instance is the stochastic Burgers modeling of turbulence where the noise is correlated in both time and space [23]. This equation is, in addition, closely related to the Kardar-Parisi-Zhang description of surface growth that was analyzed with spatially correlated noise in, e.g., [24, 25] and references therein. The so-called fractional Brownian motion [26] is actually defined by the non-Markovian nature of the process, which can be traced back to the temporal correlations characterizing the thermal noise. The physical circumstances in which a temporally correlated noise arises are manifold including polymer translocation through a nanopore [27] or the effective description of a tracer in a glassy medium [28]. A review of the effects of colored noise in dynamical systems is given in [29]. Last but not least, the environment fluctuations in quantum dissipative systems give rise rather naturally to temporally correlated contributions [18].

In the present study we explore the influence of colored noise on the equilibrium and non-equilibrium critical dynamics of the N-component ϕ^4 Ginzburg-Landau model with $O(N)$ symmetry and non-conserved order parameter. We focus on the case of noise correlations that decay as a power-law in the time-lag, such that no time scale can be associated to it at least for sufficiently long times. To our knowledge critical dynamics with such a colored noise have not been theoretically investigated yet. Our first aim is to present a general equilibrium analysis that allows us to determine the classes of noise which affect the critical relaxation. We use the perturbative renormalization group (RG) technique to calculate the critical exponents, which turn out to be modified for a certain type of noise correlations only. Our second aim is to understand the non-equilibrium dynamics in the presence of such kind of colored noise. A short time after the critical quench, the dynamics of the system can be described in terms of the non-equilibrium critical exponent θ which depends on the properties of the colored noise. In the critical coarsening regime we define an *effective temperature* [30, 31] β_∞^{-1} from the long-time limit of the so-called fluctuation-dissipation ratio [c.f., Eq. (4.40) for its definition] which was proposed to be an additional universal property of the non-equilibrium dynamics [32]. β_∞^{-1} was calculated for a variety of critical processes with white noise [14, 33, 34]. We show that β_∞^{-1} turns out to be affected by the color of the noise and we compare our results to the corresponding ones for the fractional Brownian motion [28].

The paper is organized as follows. In Sec. 2 we present the model and we obtain the associated dynamic action by using the path-integral formalism [1, 3, 2, 7, 8, 35]. We define the response and the correlation functions, we present their scaling behavior and we define the critical exponents. In Sec. 3 we focus on the equilibrium properties of the model, we calculate the propagators in the frequency and momentum domain, presenting the Feynman rules of the perturbation theory. This section contains details on the perturbative RG analysis which allows us, on the one hand, to determine the type of noise which alters the critical dynamics and, on the other, to calculate the dynamical critical exponent z up to second order in $\epsilon = D_c - D$ where D_c is the

upper critical dimensionality, which we find to be 4. In Sec. 4 we focus on the non-equilibrium dynamics. We derive the critical scaling behavior of the linear response and correlation functions and we calculate the non-equilibrium critical exponent θ (the "initial slip exponent", see [10]) up to first order in $4-D$. Using the scaling forms of the correlation and linear response functions we deduce the effective temperature after a critical quench. Finally, in Sec. 5 we summarize our results and we discuss several of their possible applications together with plans for further investigation. The details of the calculations are reported in the Appendices.

2. The model

In D spatial dimensions we consider an N-component non-conserved order parameter $\vec{\phi}(\vec{x})$ whose static behavior near the critical point is governed by a Hamiltonian of the Ginzburg-Landau type

$$\mathcal{H}[\vec{\phi}] = \int \mathrm{d}^D x \left[\frac{1}{2}(\vec{\nabla}\vec{\phi})^2 + \frac{1}{2} r\vec{\phi}^2 + \frac{g}{4!}\vec{\phi}^4\right]. \tag{2.1}$$

g is the strength of the non-linearity that drives the phase transition, r is the control parameter for it, and the coefficient in front of the elastic term $\propto (\vec{\nabla}\vec{\phi})^2$ has been absorbed in the definition of the field. The time evolution under purely dissipative dynamics (model A according to the standard classification of [7]) in the presence of memory is described by the Langevin equation

$$\int_{-T}^{t} \mathrm{d}t'\, \Gamma(t-t')\partial_{t'}\vec{\phi}(\vec{x},t') + \frac{\delta\mathcal{H}}{\delta\vec{\phi}(\vec{x},t)} = \vec{\zeta}(\vec{x},t), \tag{2.2}$$

where $-T$ is the initial time of the process and $\vec{\zeta}$ is a zero-mean Gaussian colored noise with

$$\langle\zeta_i(\vec{x},t)\zeta_j(\vec{x}',t')\rangle = \beta^{-1}\Gamma(t-t')\delta(\vec{x}-\vec{x}')\delta_{ij}. \tag{2.3}$$

Γ is a positive and symmetric kernel, i.e., $\Gamma(t-t') = \Gamma(t'-t) > 0$. Depending on the specific system of interest, in the minimal dynamical model in Eq. (2.2) one might need to include an inertial term $M\partial_t^2\phi$. An order of magnitude analysis suggests that such a term is relevant for times shorter than $M/\int \mathrm{d}t\ \Gamma(t)$, whereas the effects of inertia can be neglected for longer times, indeed the regime we are interested in. For this reason we shall omit this term from the outset as long as the contribution of friction does not vanish. Note that the function Γ determines both the noise-noise correlation [Eq. (2.3)] and the time-dependent retarded friction coefficient [Eq. (2.2)] since we have assumed the thermal bath (which is weakly coupled to the system) to be in equilibrium at temperature β^{-1} [we set $k_B = 1$]. The Markovian examples of this dynamics are characterized by a δ-correlated ('white') noise, i.e.,

$$\Gamma(t) = 2\gamma_{\mathrm{w}}\delta(t), \tag{2.4}$$

where γ_{w} is the friction coefficient. In this case Eq. (2.2) has been extensively studied both in and out of equilibrium, see, e.g., [3, 10, 14, 36]. Such *Ohmic* dissipation is the

simplest form of *short-range correlated* noise. It can be formally obtained as the limit $t_0 \to 0$ of the exponentially correlated Ornstein-Uhlenbeck (OU) process

$$\Gamma_{\mathrm{OU}}(t) = \frac{\gamma_{\mathrm{OU}}}{t_0} \mathrm{e}^{-|t|/t_0}, \tag{2.5}$$

where the finite characteristic relaxation time t_0 plays the role of an internal scale. Under renormalization one expects the exponentially correlated noise to become equivalent to a white (delta-correlated) one, Eq. (2.4), and the critical behavior of the OU process be identical to the Markovian one. In the absence of an internal scale, instead, there is no reason to expect a white noise limit and the critical behavior might be affected. The simplest example with no explicit time scale is

$$\Gamma(t) = \frac{\gamma}{\Gamma_E(1-\alpha)}\ |t|^{-\alpha} \qquad \text{with } \ \alpha > 0. \tag{2.6}$$

We denote by Γ_E the Euler Γ-function to avoid confusion with the noise kernel Γ. For $\alpha > 1$, i.e., *super Ohmic dissipation*, expression (2.6) is not integrable, unless a short-time cut-off and thus an internal scale is introduced. One can show that under naive scaling (introduced in Sec. 3) the Fourier or Laplace transform of $\Gamma(t)$ generate a white noise vertex that dominates over the colored noise part. Hence, the appearance of a cut-off scale suggests the non relevance of the colored noise for *super-Ohmic* dissipation, i.e., $\alpha > 1$. This statement will be made precise in the following. Instead, for *sub-Ohmic dissipation*, i.e., $\alpha < 1$, the noise is truly long-range correlated and its influence on the dynamics will turn out to be non-trivial. The naive cross-over value between these two cases is $\alpha = \alpha_c = 1$, that is white noise or *Ohmic dissipation*. In the presence of interactions we shall show that this scenario is slightly modified, with the cross-over value $\alpha_c(D, N)$ depending upon D and N.

A functional-integral representation of the stochastic process, Markovian or not, is better suited for an analytic treatment of critical dynamics than the Langevin equation (2.2). In particular, it allows one to express the average $\langle \cdots \rangle_\zeta$ over the possible realizations of the noise $\vec{\zeta}$ in Eq. (2.2) as a functional integral (which will be denoted by $\langle \cdots \rangle$ in what follows)

$$\langle \cdots \rangle_\zeta = \int [\mathrm{d}\phi \mathrm{d}\bar{\phi}]\ \cdots\ \mathrm{e}^{-\mathcal{S}[\phi,\bar{\phi}]} \tag{2.7}$$

over ϕ and an auxiliary field $\bar{\phi}$ with $\mathcal{S} = \mathcal{S}_0 + \mathcal{S}_{int} - \ln \mathcal{P}_{IC}$ [1, 2, 3, 35]‡,

$$\begin{aligned} \mathcal{S}_0 = &\int \mathrm{d}^D x \int_{-T}^{\infty} \mathrm{d}t\ \overline{\phi}_i(\vec{x}, t) \left[\int_{-T}^{t} \mathrm{d}t'\ \Gamma(t-t')\partial_{t'}\phi_i(\vec{x}, t') + (r - \nabla^2)\phi_i(\vec{x}, t) \right] \\ &- \beta^{-1} \int \mathrm{d}^D x \int_{-T}^{\infty} \mathrm{d}t \int_{-T}^{t} \mathrm{d}t'\ \overline{\phi}_i(\vec{x}, t)\Gamma(t-t')\overline{\phi}_i(\vec{x}, t') \end{aligned} \tag{2.8}$$

and

$$\mathcal{S}_{int} = \int \mathrm{d}^D x \int_{-T}^{\infty} \mathrm{d}t\ \frac{g}{3!}\overline{\phi}_i(\vec{x}, t)\phi_i(\vec{x}, t)\phi_j(\vec{x}, t)\phi_j(\vec{x}, t). \tag{2.9}$$

‡ In the presence of colored noise no discretization problems arise, see, e.g., [35].

We used Einstein's convention of summation over repeated indices. The zero-source functional integral is identical to 1 due to the normalization of the noise probability distribution. $\mathcal{P}_{IC}[\vec{\phi}(\vec{x},-T)]$ is the statistical weight of the initial condition. The auxiliary field $\vec{\overline{\phi}}$§ is conjugated to an external perturbation $\vec{h}$, in such a way that if $\mathcal{H}[\vec{\phi},\vec{h}] = \mathcal{H}[\vec{\phi}] - \vec{\phi}\cdot\vec{h}$, the linear response of the order parameter to the field $\vec{h}$ is given by

$$R(\vec{x}-\vec{x}';t,t')\delta_{ij} = \left.\frac{\delta\langle\phi_i(\vec{x},t)\rangle_{\vec{h}}}{\delta h_j(\vec{x}',t')}\right|_{\vec{h}=\vec{0}} = \langle\phi_i(\vec{x},t)\overline{\phi}_j(\vec{x}',t')\rangle, \tag{2.10}$$

where $\langle\cdots\rangle_{\vec{h}}$ is the average over the stochastic process in the presence of the external perturbation, i.e., Eq. (2.2) with $\mathcal{H} \mapsto \mathcal{H}[\vec{\phi},\vec{h}]$. The response function is causal irrespectively of the noise statistics and the Jacobian of the transformation of variables from $\vec{\zeta}$ to $\vec{\phi}$ which allows us to write the average over the stochastic process as in Eq. (2.7) is also a factor with no consequences [35]. In addition to the (linear) response function, we shall consider below the correlation function of the order parameter, defined by

$$C(\vec{x}-\vec{x}',t,t')\delta_{ij} = \langle\phi_i(\vec{x},t)\phi_j(\vec{x}',t')\rangle \tag{2.11}$$

where we assumed translational invariance in space. The action $\mathcal{S}_0 + \mathcal{S}_{int}$ is the sum of two contributions each one made of several terms. The part with density $\bar{\phi}_i\delta\mathcal{H}/\delta\phi_i$ represents the deterministic dynamics whereas the remaining part is due to the coupling to the bath. The latter consists of the friction term and the noise-noise correlation and both involve the kernel Γ. In this formulation the problem is recast in the form of a field theory in $D+1$ dimensions with two vector fields, the analysis of which can be done via standard field-theoretical tools, such as the renormalization group (RG) approach that we shall use below.

Since, in general, there is no tractable Fokker-Planck equation for the non-Markov stochastic processes we are presently interested in, the usual and relatively simple proof of equilibration explained in, e.g., [37, 38] for the white-noise problem does not apply. However, we recall here that Eq. (2.2) is an effective description of the dynamics of a classical system with Hamiltonian $\mathcal{H}'$ which is weakly and linearly coupled to a (large) equilibrium bath of harmonic oscillators at temperature β^{-1}, acting as a source of the stochastic noise $\vec{\zeta}$ effectively induced by such a coupling. Indeed, the temperature that characterizes the correlations of the noise in Eq. (2.3) is β^{-1}, whereas the distribution of the frequencies of the harmonic oscillators within the bath determines the functional form of Γ. In addition, Γ appears in Eq. (2.2) and Eq. (2.3) in such a way to ensure the fluctuation-dissipation theorem for the bath variables [see, c.f., Eq. (2.12)]. As a result, even with this effective non-Markov dynamics the system should still lose memory of its initial condition and equilibrate with the equilibrium bath of oscillators, resulting in a canonical distribution $\mathrm{e}^{-\beta\mathcal{H}[\vec{\phi}]}/\mathcal{Z}(\beta)$ of one-time quantities at sufficiently long times (possibly divergent with the system size) where $\mathcal{Z}(\beta)$ is the partition function and $\mathcal{H}$ differs from $\mathcal{H}'$ by a term which is quadratic in the relevant degrees of freedom (see,

§ $\vec{\overline{\phi}}$ is purely imaginary and it is sometimes written as $i\vec{\overline{\phi}}$ in the literature.

e.g., [18, 39] for details). The asymptotic critical *equilibrium dynamics* is expected to be described by the limit $T \to \infty$ of the action in which one neglects the specific distribution $\mathcal{P}_{IC}$ of the initial conditions that in any case should be forgotten dynamically. Since we shall be interested in the critical dynamics, we set $\beta = \beta_c$ and we absorb this constant into a redefinition of the fields and of the coupling constant g. In equilibrium the response and the correlation functions are invariant under time translations, i.e., $R(\vec{x}, t, t') = R(\vec{x}, t - t')$ [see Eq. (2.10)] and $C(\vec{x}, t, t') = C(\vec{x}, t - t')$ [Eq. (2.11)] , and they are related to each other by the fluctuation-dissipation theorem (FDT) that reads $R(\vec{x}, t) = -\beta \partial_t C(\vec{x}, t)\Theta(t)$, where t represents the time delay, $\Theta(t \le 0) = 0$ and $\Theta(t > 0) = 1$ ‖, and which is completely independent of the specific characteristics of the system and the bath apart from its temperature. (A proof of this relation for generic colored noise Langevin dynamics can be found in [35].) Once the latter has been absorbed in the redefinition of ϕ_i and g the FDT becomes

$$R(\vec{x}, t) = -\partial_t C(\vec{x}, t)\Theta(t), \tag{2.12}$$

and this is the form that we shall use in our calculations. Moreover, the time-dependent correlation is invariant under time-reversal, i.e., $C(\vec{x}, t) = C(\vec{x}, -t)$.

Non-equilibrium dynamics, instead, can be studied by leaving T finite and by making the initial distribution $\mathcal{P}_{IC}$ explicit [10, 14]. A typical choice is a Gaussian weight in which case β_c can still be absorbed into a redefinition of the fields and g. Stationarity is lost out of equilibrium and correlation and linear response functions depend on all times involved in their definitions (t and t' in Eqs. (2.10) and (2.11)). Moreover, the FDT is no longer valid [14, 33, 34].

In addition to R defined in Eq. (2.10) and C defined in Eq. (2.11), one can construct the quadratic correlator $\langle \overline{\phi}_i(\vec{x}, t)\overline{\phi}_j(\vec{x}', t')\rangle$ which, independently of the color of the noise, vanishes identically due to causality.

2.1. Scaling

In the case of stochastic dynamics with white noise, a systematic RG analysis confirms the phenomenological scaling behavior of the linear response and correlation functions both for $T \to \infty$ and T finite corresponding, respectively, to equilibrium and non-equilibrium relaxational dynamics. In terms of the equilibrium correlation length $\xi_{\rm eq} \simeq |r - r_c|^{-\nu}$, where r_c is the critical value of the parameter r in Eq. (2.1), and of a dynamic growing length $\xi(t) \simeq t^{1/z}$, one expects [10, 14]

$$R(\vec{p}, t, t') = p^{-2+\eta+z}\ [\xi(t)/\xi(t')]^{z\theta}\ f_R(p\xi_{\rm eq}, \xi(t)/\xi_{\rm eq}, \xi(t')/\xi(t)) \tag{2.13}$$

and

$$C(\vec{p}, t, t') = p^{-2+\eta}\ [\xi(t)/\xi(t')]^{z(\theta-\hat{\alpha})}\ f_C(p\xi_{\rm eq}, \xi(t)/\xi_{\rm eq}, \xi(t')/\xi(t)), \tag{2.14}$$

for the Fourier transform in space of $R(\vec{x}, t, t')$ and $C(\vec{x}, t, t')$, respectively, with the white noise value $\hat{\alpha} = 1$ [10]. In the previous expressions, ν is the standard static critical

‖ Note that the Itô prescription of the Langevin equation (2.2) implies $\Theta(0) = 0$ in the stochastic path integral description [37, 3].

exponent associated with the correlation length, z is the dynamic critical exponent which characterizes the different scaling behavior of space and time, whereas η is the static anomalous dimension of the field ϕ and it controls the power-law spatial decay of the static correlation function. θ is the so-called initial-slip exponent [10, 13, 14] that accounts for the effects of the initial condition in the case of finite T. It is a novel universal quantity if the relaxation occurs from a disordered initial state, whereas it is related to known equilibrium exponents if the initial state has a non-vanishing average value of the order parameter [11, 12]. In Eq. (2.13) and Eq. (2.14) $f_{R,C}$ are scaling functions which become universal after the introduction of proper normalization. Equilibrium dynamical scaling is recovered in the limiting case $\xi(t') \simeq \xi(t) \gg \xi_{\rm eq}$ (i.e., in the limit of long times t, t' with finite $t - t'$), whereas aging phenomena are expected to emerge for $\xi(t), \xi(t') \ll \xi_{\rm eq}$ and, in particular, right at the critical point $r = r_c$. In the presence of specific instances of correlated noise we expect the scaling behavior in Eq. (2.13) and Eq. (2.14) to be modified both as far as the exponents and the scaling functions are concerned. The changes appear at the level of the Gaussian theory and non-trivial effects survive in the presence of interactions for certain noise correlations, as we shall explain in Secs. 3 and 4.

3. Equilibrium dynamics

According to the interpretation of the Langevin dynamics in Eq. (2.2) as resulting from the coupling to an equilibrium thermal bath, after a sufficiently long time the system is expected to relax towards an equilibrium state characterized by the effective Hamiltonian $\mathcal{H}$, i.e., by the static ϕ^4-theory. This relaxation occurs generically and for arbitrary initial conditions as long as the asymptotic values of the control parameters of the system (r in the case we are concerned with) imply for $\mathcal{H}$ neither a spontaneous symmetry breaking nor criticality, which would indeed provide instances of *aging* (see, e.g., [39]). However, the existence of a wide region of parameter space ($r > 0$) for which equilibration occurs, allows us to conclude that all static properties of a theory with effective Hamiltonian $\mathcal{H}$ carry over to the dynamic field-theoretical action $\mathcal{S}$ [see Eq. (2.8) and Eq. (2.9)] which generates the dynamic correlation functions and therefore the static ones as a special case. The upper critical dimensionality D_c above which the Gaussian theory becomes exact is therefore the same as in the ϕ^4 theory, i.e., $D_c = 4$ (see, e.g., [37]). Analogously, the same applies to the static exponents ν and η. In this section we show how this arises within perturbation theory. In particular, we determine the conditions under which the critical dynamics is modified by the colored part of the noise with special focus on the emergence of a cross-over line $\alpha_c(D, N)$ which bounds the region within which the dynamic exponent z is affected by the color of the noise. We calculate this exponent in the white and colored noise cases.

3.1. Gaussian theory

In the $T \to \infty$ limit the Gaussian part of the action $\mathcal{S}_0$ can be diagonalized via a Fourier transform of the fields defined in Eq. (A.1). One obtains

$$\mathcal{S}_0 = \frac{1}{2} \int \frac{\mathrm{d}\omega}{2\pi} \frac{\mathrm{d}\omega'}{2\pi} \int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{\mathrm{d}^D p'}{(2\pi)^D} \, \vec{\varphi}^T(\vec{p}, \omega) \mathcal{C}(\vec{p}, \omega; \vec{p'}, \omega') \vec{\varphi}(\vec{p'}, \omega'), \tag{3.1}$$

where we used a vector notation $\vec{\varphi} = (\vec{\phi}(\vec{p}, \omega)\, ,\, \vec{\overline{\phi}}(\vec{p}, \omega))^T$ for the $2N$-component field $\vec{\varphi}$ and we introduced the correlation matrix

$$\mathcal{C} = \delta_{ij} \delta(\vec{p} + \vec{p'}) \delta(\omega + \omega') \begin{pmatrix} 0 & i\omega\Gamma_{i\omega} + (p^2 + r) \\ -i\omega\Gamma^*_{i\omega} + (p^2 + r) & -(\Gamma_{i\omega} + \Gamma^*_{i\omega}) \end{pmatrix}. \tag{3.2}$$

Here and in what follows we denote a function and its Fourier transform with the same symbol, the difference being made clear by their arguments. In Eq. (3.2) $\Gamma_{i\omega}$ stands for the Fourier transform of $\Theta(t)\Gamma(t)$ [the $\Theta(t)$ factor is a consequence of the causal structure of Eq. (2.2)]. As usual, * denotes the complex conjugate. For the colored noise in Eq. (2.6) one finds

$$\Gamma_{i\omega} = \gamma |\omega|^{\alpha-1} \left[\sin(\pi\alpha/2) - i \,\mathrm{sign}(\omega) \cos(\pi\alpha/2) \right] + \gamma_{\mathrm{w}}. \tag{3.3}$$

[Note that for $\alpha > 1$ a short-time cut-off has to be introduced in order to transform Eq. (2.6). However, the dynamic properties we are presently interested in are determined by the leading behavior at small ω, which is not affected by the introduction of such a cut-off and is correctly captured by Eq. (3.3). Accordingly, we shall use this form irrespectively of the value of α.] In this expression we have added a supplementary *white-noise vertex* γ_{w} for reasons that will become clear in the following [note that the cut-off that has to be introduced in order to make Eq. (2.6) integrable for $\alpha > 1$ effectively leads to this supplementary white-noise vertex]. The propagators¶ are deduced by inverting $\mathcal{C}$:

$$R_0(\vec{p}, \omega) \delta_{ij} = \langle \phi_i(\vec{p}, \omega) \overline{\phi}_j(-\vec{p}, -\omega) \rangle = \frac{1}{i\omega\Gamma_{i\omega} + p^2 + r} \, \delta_{ij} \tag{3.4}$$

and

$$\begin{aligned} C_0(\vec{p}, \omega) \delta_{ij} &= \langle \phi_i(\vec{p}, \omega) \phi_j(-\vec{p}, -\omega) \rangle \\ &= \frac{\Gamma_{i\omega} + \Gamma^*_{i\omega}}{\omega^2 \Gamma_{i\omega} \Gamma^*_{i\omega} + i\omega(p^2 + r)(\Gamma_{i\omega} - \Gamma^*_{i\omega}) + (p^2 + r)^2} \, \delta_{ij}. \end{aligned} \tag{3.5}$$

By construction they satisfy the FDT [see Eq. (2.12)] that in the frequency domain reads:

$$2i \,\mathrm{Im}\, R_0(\vec{p}, \omega) = -i\omega C_0(\vec{p}, \omega). \tag{3.6}$$

We recall that we absorbed the temperature β^{-1} in a redefinition of the fields and the coupling constant g, and that $C_0(\vec{p}, \omega)$ is a real function.

¶ In what follows we denote the response and correlation function by R and C, respectively. The various propagators and quantities within the Gaussian approximation are denoted by the subscript $_0$.

The static correlation function $C_0(\vec{p}, t = 0)$ can be obtained by integrating Eq. (3.5) over the frequency ω and, as expected, the result agrees with the static Gaussian correlation that one would infer from the Hamiltonian $\mathcal{H}$ [see, c.f., the calculation leading to Eq. (B.4)]. Consequently, the static critical exponents ν and η are not modified at this order by the dynamics and they take the Gaussian values $\nu_0 = 1/2$ and $\eta_0 = 0$, respectively.

We anticipate here that in Sec. 4.1, while discussing the non-equilibrium dynamics of the present model, we consider the Laplace transform [see Eq. (A.2)] of Eq. (2.2) with $g = 0$ and the colored noise given in Eq. (2.6) (i.e., with $\gamma_{\rm w} = 0$). This allows us to determine the Laplace transform of the response function R_0, formally obtained by replacing $i\omega$ with λ in Eq. (3.4); compare Eqs. (A.1) and (A.2). This transform can be inverted to a form given in terms of the so-called generalized Mittag-Leffler functions $E_{\alpha,\beta}$ defined in Eq. (B.18) and provides a closed expression for $R_0(\vec{p}, t)$:

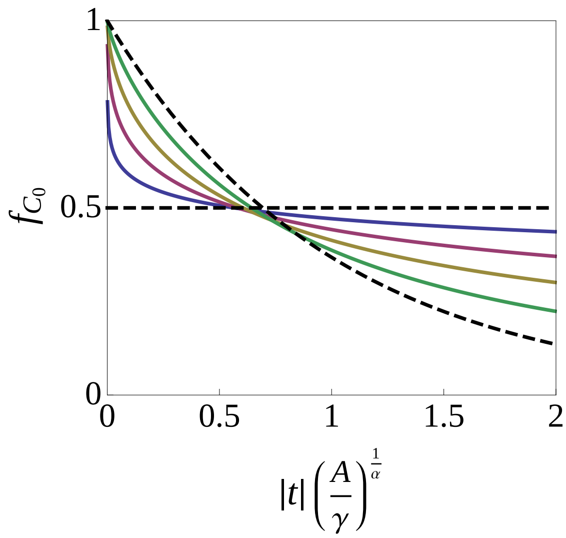
$$R_0(\vec{p}, t) = \Theta(t) \frac{t^{\alpha-1}}{\gamma} E_{\alpha,\alpha}(-A t^\alpha/\gamma), \tag{3.7}$$

where $A \equiv p^2 + r$. The equilibrium correlation function C_0 is readily determined from this expression via the fluctuation-dissipation theorem Eq. (2.12) (see, c.f., App. B.2 for details):

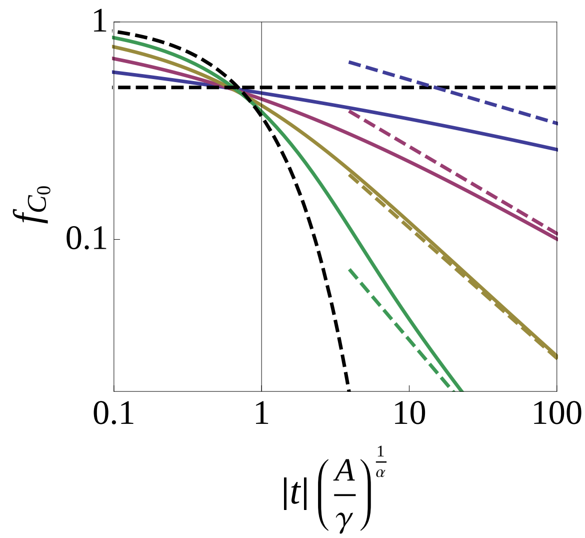
$$C_0(\vec{p}, t) = \frac{1}{A} E_\alpha(-A|t|^\alpha/\gamma) \tag{3.8}$$

where $E_\alpha(z) \equiv E_{\alpha,1}(z)$. In Fig. 1(a) we plot AC_0 as a function of the (dimensionless) scaling variable $u \equiv |t|(A/\gamma)^{1/\alpha}$ associated with time t. For $\alpha \to 1$ one recovers the purely exponential dependence e^{-u} (indicated by the decreasing dashed curve in Fig. 1) which characterizes the case of white noise. As α decreases, instead, the correlation function displays a faster initial drop followed by a slower decay at large values of u. Indeed, taking into account the known asymptotic behavior of the Mittag-Leffler functions [c.f., Eq. (B.19)], these curves decay algebraically as $\sim 1/[\Gamma_E(1-\alpha)u^\alpha]$ for $u \to \infty$. In panel (b) of Fig. 1 we use a log-log-scale to compare the curves shown in panel (a) with their corresponding leading asymptotic algebraic decays, indicated by the straight dashed curves for $u \gtrsim 5$. As $\alpha \to 0$ the approximation provided by the leading term of the asymptotic expansion becomes less accurate in this time span and one needs to go to longer times to reach the asymptotic regime. The curves in Fig. 1 clearly illustrate the crossover between an exponential and an algebraic asymptotic behavior of the correlation function as α decreases below the value $\alpha = 1$.

For the generic case of the noise in Eq. (3.3) with $\gamma, \gamma_{\rm w} \neq 0$, the propagators R_0 and C_0 do not have a simple analytic form in the time domain, in contrast to the familiar exponential relaxation which characterizes the case with white noise ($\gamma = 0$, $\gamma_{\rm w} \neq 0$) briefly recalled in Eqs. (B.1) and (B.2) and to the purely colored case discussed in the previous paragraph ($\gamma \neq 0$ and $\gamma_{\rm w} = 0$). In spite of this difficulty, the Gaussian value z_0 of the dynamic exponent z can be determined by comparing the scaling of the first two terms in the denominator of R_0 for small ω and p since one expects $\omega \sim p^z$ from the definition of z (see, e.g., [37]). First we note that for small ω, Eq. (3.3) scales as



(a)



(b)

Figure 1. Scaling function AC_0 of the Gaussian correlation C_0 in equilibrium ($T \to \infty$) as a function of $u \equiv |t|(A/\gamma)^{1/\alpha}$ for various values of α, with $A = \vec{p}^{\,2} + r$. (a) The horizontal dashed line corresponds to the limit $\alpha \to 0$, whereas the other dashed line points out the purely exponential behavior e^{-u}, which is recovered for $\alpha = 1$. The solid lines, instead, correspond to $\alpha = 0.2$, 0.4, 0.6, and 0.8, from bottom to top at small u and in the reverse order at large u. (b) Log-log plot of the curves shown in panel (a) compared to their corresponding leading asymptotic algebraic behavior inferred from (B.19), which are indicated as (straight) dashed lines.

$\Gamma_{i\omega} \sim |\omega|^{\alpha-1}$ for $\alpha < 1$, whereas $\Gamma_{i\omega} \sim 1$ for $\alpha > 1$: in the former case the effect of the colored part of the vertex is dominant, whereas in the latter the contribution of the white noise ($\propto \gamma_{\mathrm{w}}$) dominates. As a result, from the scaling $\omega\Gamma_{i\omega} \sim p^2$ one can read the Gaussian value z_0 of the dynamic exponent:

$$z_0 = \begin{cases} z_0^{(\mathrm{col})} = 2/\alpha & \text{for } \alpha < 1, \\ z_0^{(\mathrm{w})} = 2 & \text{for } \alpha \geq 1. \end{cases} \tag{3.9}$$

A similar effect is observed in diffusion processes with colored noise, the so-called fractional Brownian motion [26]. The particle's displacement is stationary and characterized by an α–dependent exponent which is called *Hurst exponent* in this context.

By rescaling the momentum p and frequency ω according to $p \mapsto b^{-1}p$ and $\omega \mapsto b^{-z}\omega$ with b the *scaling parameter* of the RG flow, one deduces the Gaussian scaling behavior of the response and the correlation propagator. We infer from Eqs. (3.4) and (3.5) that

$$b^{-2}R_0(b^{-1}\vec{p}, b^{-z_0}\omega; r, \gamma, \gamma_{\mathrm{w}}) = R_0(\vec{p}, \omega; b^2 r, b^{2-\alpha z_0}\gamma, b^{2-z_0}\gamma_{\mathrm{w}}), \tag{3.10}$$

with a similar expression for C_0, where the prefactor b^{-2} on the left-hand side (lhs) is replaced by b^{-2-z}. As anticipated, one can identify two asymptotically scale-invariant behaviors (the so-called Gaussian fixed-points in the parameter space) as the Gaussian

critical point $r = 0$ is approached. They correspond to $P \equiv (\gamma_{\rm w} = 0, \gamma \neq 0)$ for $\alpha < 1$ and $P_{\rm w} \equiv (\gamma_{\rm w} \neq 0, \gamma = 0)$ for $\alpha \geq 1$, i.e., to the cases in which either the colored or the white noise is relevant. The latter reduces to the standard Model A dynamics [37]. In order for P and $P_{\rm w}$ to be fixed points, it is necessary that the corresponding non-vanishing coupling strengths, either γ or $\gamma_{\rm w}$, are constant under renormalization which, as expected from Eq. (3.9), implies $z = z_0^{(\rm col)} = 2/\alpha$ for $\alpha < 1$ (P) and $z = z_0^{(\rm w)} = 2$ for $\alpha \geq 1$ ($P_{\rm w}$).

In order for the action $\mathcal{S}_0$ to be invariant under the momentum and frequency rescaling discussed above, one has to rescale the fields ϕ_i and $\overline{\phi}_i$ as $\phi_i(b^{-1}\vec{p}, b^{-z_0}\omega) \mapsto b^{d_\phi}\phi_i(\vec{p},\omega)$ and $\overline{\phi}_i(b^{-1}\vec{p}, b^{-z_0}\omega) \mapsto b^{d_{\overline{\phi}}}\overline{\phi}_i(\vec{p},\omega)$ where d_ϕ and $d_{\overline{\phi}}$ are the so-called scaling dimensions of the fields $\vec{\phi}$ and $\vec{\overline{\phi}}$, respectively, in the $(\vec{p},\omega)$-domain. (Below we shall introduce the scaling dimensions of the fields in the time-domain; in order to keep the notation as simple as possible we do not include an additional subscript to distinguish the two cases but we explain in the text which one we use in each case.) The latter take the Gaussian values

$$d_{\phi,0} = (D+2)/2 + z_0, \tag{3.11}$$

$$d_{\overline{\phi},0} = (D+2)/2. \tag{3.12}$$

In the white-noise case $z_0 = 2$ we recover the standard scaling dimensions of Model A critical dynamics [37]. As far as the transformation properties of the propagators under these rescalings are concerned we have $b^{-2d_\phi+D+z_0}C_0(b^{-1}\vec{p}, b^{-z_0}\omega;\ldots) = C_0(\vec{p},\omega;\ldots)$ and $b^{-d_\phi-d_{\overline{\phi}}+D+z_0}R_0(b^{-1}\vec{p}, b^{-z_0}\omega;\ldots) = R_0(\vec{p},\omega;\ldots)$ where the factor b^{D+z_0} comes from the δ-function which guarantees the conservation of momenta and frequencies. We have not specified the scaling of the parameters r, γ and $\gamma_{\rm w}$ to lighten the notation. By comparing with the scaling behavior of the Gaussian response in Eq. (3.10) and of the correlation function, one confirms the Gaussian values (3.11) and (3.12) for the dimensions d_ϕ and $d_{\bar{\phi}}$, respectively.

In Eq. (3.3) we added to the colored-noise vertex associated with Eq. (2.6) a white-noise contribution proportional to $\gamma_{\rm w}$ for the purpose of highlighting the emergence of the two distinct Gaussian fixed points P and $P_{\rm w}$. As we shall show below such a white-noise contribution is anyhow generated under the RG flow as soon as one accounts for the effect of non-Gaussian fluctuations (i.e., $g \neq 0$) on the Gaussian fixed-point $P = (\gamma \neq 0, \gamma_{\rm w} = 0)$ with colored noise alone.

3.2. The interaction part

The interaction part of the action reads

$$\mathcal{S}_{int} = \int \frac{{\rm d}\omega}{2\pi}\frac{{\rm d}\omega'}{2\pi}\frac{{\rm d}\omega''}{2\pi}\frac{{\rm d}^Dp}{(2\pi)^D}\frac{{\rm d}^Dp'}{(2\pi)^D}\frac{{\rm d}^Dp''}{(2\pi)^D}\,\frac{g}{3!}\overline{\phi}(-\vec{p}-\vec{p'}-\vec{p''}, -\omega-\omega'-\omega'') \times \phi(\vec{p},\omega)\phi(\vec{p'},\omega')\phi(\vec{p''},\omega'')$$

in the frequency and momentum domain. Under the naive scaling with Eqs. (3.9), (3.11), and (3.12) one easily obtains the scaling of the coupling constant: $g \to b^{4-D}g$.

The upper critical dimension is thus $D_c = 4$ independently of α and the effects of fluctuations beyond mean-field can be accounted for by using a standard perturbative expansion in terms of $\epsilon = 4 - D$.

In the presence of the interaction $\mathcal{S}_{int}$, the scaling dimension of the fields and the coupling constants are altered. In addition, we shall show that the crossover value $\alpha_c = 1$, which separates the colored-noise-dominated case from the white-noise-dominated one, acquires a dependance on D, thus dividing the (α, D)-plane (for N fixed) in two distinct regions, each one with different scaling properties. Under a RG flow with scaling parameter $b > 1$ the noise strengths γ and $\gamma_{\rm w}$ scale as

$$\gamma \ \mapsto b^{2-\alpha z_0+\alpha\eta_\gamma}\ \gamma, \tag{3.13}$$

$$\gamma_{\rm w} \mapsto b^{2-z_0+\eta_{\rm w}}\ \gamma_{\rm w}, \tag{3.14}$$

which generalize the corresponding Gaussian scaling behavior of these parameters — encoded in Eq. (3.10) — via the introduction of suitable anomalous dimensions η_γ and $\eta_{\rm w}$ of γ and $\gamma_{\rm w}$, respectively. These anomalous dimensions η_γ and $\eta_{\rm w}$ determine the corrections to the Gaussian value z_0 of the dynamical exponent z and the crossover value α_c which separates the different regions in the (α, D)-plane. Indeed, let l be a length scale and τ be a time scale. Dimensional analysis implies $t \sim \tau$ and $x \sim l$. From Eq. (3.4) we infer that $\gamma \sim \tau^\alpha/l^2$ and $\gamma_{\rm w} \sim \tau/l^2$. Consider the case in which the colored noise dominates, which corresponds to having $2 - \alpha z_0 + \alpha\eta_\gamma > 2 - z_0 + \eta_{\rm w}$ in terms of the dimensions of the noise strengths [see Eqs. (3.13) and (3.14)] with $z_0 = 2/\alpha$. By choosing $\tau^\alpha = l^2\gamma$ we have $t \sim l^{2/\alpha}\gamma^{1/\alpha}$. Therefore, under an RG flow with $l \mapsto bl$ $(b > 1)$ we deduce from Eq. (3.13) that $t \sim b^{2/\alpha+\eta_\gamma}\ l^{2/\alpha}\gamma^{1/\alpha}$. On the other hand, by noting that the dynamic exponent z is defined through $t \to b^z t$ we can readily identify the dynamic exponent $z = 2/\alpha + \eta_\gamma$ in terms of η_γ. In the white-noise-dominated case we choose $\tau = l^2\gamma_{\rm w}$ and a similar argument yields the white-noise result $z = 2 + \eta_{\rm w}$. In short,

$$z = \begin{cases} 2 + \eta_{\rm w} & \text{for } \ \alpha > \alpha_c(D, N), \\ 2/\alpha + \eta_\gamma & \text{for } \ \alpha < \alpha_c(D, N), \end{cases} \tag{3.15}$$

and therefore one needs to calculate $\eta_{\rm w}$ and η_γ in order to determine z.

In the presence of non-Gaussian fluctuations, the scaling dimensions $d_\phi = d_{0,\phi} - z_0 - \eta/2$ and $d_{\overline{\phi}} = d_{0,\overline{\phi}} - z_0 - \overline{\eta}/2$ in the $(\vec{p}, t)$-domain of the fields ϕ and $\bar{\phi}$, respectively, differ from their Gaussian values by the corresponding anomalous dimensions η and $\overline{\eta}$ (the extra $-z_0$ comes from the conversion of $d_{0,\phi}$ and $d_{0,\overline{\phi}}$ from the frequency to the time domain). In order to determine the resulting scaling in the $(\vec{p}, \omega)$-domain one has to take into account the integral over time that carries a dimension z (which differs from the Gaussian value z_0); therefore

$$\phi_i(b^{-1}\vec{p}, b^{-z}\omega) \mapsto b^{d_\phi+z-z_0}\ \phi_i(\vec{p}, \omega) = b^{D/2+1+z-\eta/2}\ \phi_i(\vec{p}, \omega), \tag{3.16}$$

$$\overline{\phi}_i(b^{-1}\vec{p}, b^{-z}\omega) \mapsto b^{d_{\overline{\phi}}+z-z_0}\ \overline{\phi}_i(\vec{p}, \omega) = b^{D/2+1+z-z_0-\overline{\eta}/2}\ \phi_i(\vec{p}, \omega). \tag{3.17}$$

The FDT implies a relation between η_γ, $\eta_{\rm w}$, η and $\overline{\eta}$, which allows one to express z in terms of the latter two. Indeed, the right-hand side (rhs) and the lhs of Eq. (2.12) should

have the same scaling dimensions; therefore $z = d_\phi - d_{\overline{\phi}}$ in terms of the dimensions of the fields in the time-domain. Using now the expressions of the field anomalous dimensions provided above, transforming into the dimensions in the frequency domain, and replacing the Gaussian values in Eq. (3.11) and Eq. (3.12) one concludes that

$$z = z_0 + \frac{\overline{\eta} - \eta}{2}. \tag{3.18}$$

3.3. Perturbative expansion

As we explained above, one does not expect any modification of equal-time correlation functions, as they are determined by a static theory with the effective Hamiltonian $\mathcal{H}$ in Eq. (2.1). Hence, we focus on the dynamical exponent z, the corrections to which can be obtained on the basis of the standard perturbative method consisting in a combined expansion in the coupling constant g and in the deviation $\epsilon = 4 - D$ from the upper critical dimensionality of the model [38, 37, 40, 41]. In performing such an expansion one also takes advantage of the fact that g will eventually be set to its fixed-point value $g^* = \mathcal{O}(\epsilon)$. We remind here that the inverse temperature β has been eliminated by a suitable redefinition of the fields and the coupling constant g. In the following we concentrate on the one-particle irreducible vertex functions [37, 41] with n external ϕ-lines and $\overline{n}$ external $\overline{\phi}$-lines, denoted[+] by

$$\mathcal{V}^{n,\overline{n}} = \mathcal{V}_0^{n,\overline{n}} + \mathcal{V}_1^{n,\overline{n}} + \mathcal{V}_2^{n,\overline{n}} + \cdots \tag{3.19}$$

The subscripts indicate the order in the perturbation series. For example, $\mathcal{V}_2^{n,\overline{n}}$ includes all terms proportional to g^2, $g\epsilon$ and ϵ^2. The Feynman rules of this perturbative expansion are those associated with the statistical weight $\mathrm{e}^{-\mathcal{S}}$ in Eq. (2.7) and they are the same as in the white noise case [10, 37], the only difference being in the form of the Gaussian response and correlation functions. In the diagrammatic representation of the perturbation series we shall indicate the relevant propagators and vertices as depicted in Fig. 2. Note that the noise vertex $\Gamma_{i\omega} + \Gamma^*_{i\omega}$ [see Fig. 2(d)] is diagonal in frequency space (i.e., it amounts to a multiplication by an ω-dependent factor) whereas it is non-local in the time domain. In addition, we point out the fact that in principle the correlation function can be obtained in the frequency domain as a multiplication of two response functions by the noise vertex, which corresponds to a convolution in the time domain.

3.3.1. Renormalization of the noise vertex. Our interest here is to know whether the correlated noise modifies the critical behavior of the model. Within the Gaussian approximation z is given by Eq. (3.9), where we assumed that a white-noise vertex is generated under renormalization, a fact that yields two distinct fixed points P and P_w: the former is characterized by the colored noise and is stable for $\alpha < \alpha_c \equiv 1$, whereas the latter is characterized by the white noise, is stable for $\alpha > \alpha_c$, and it reduces to the standard Model A dynamics. We shall show that, on the one hand, expanding around

[+] Our notation differs from the standard one, that is $\Gamma^{n,\overline{n}}$ for the 1PI-vertex functions, in order to avoid confusion with the noise kernel.

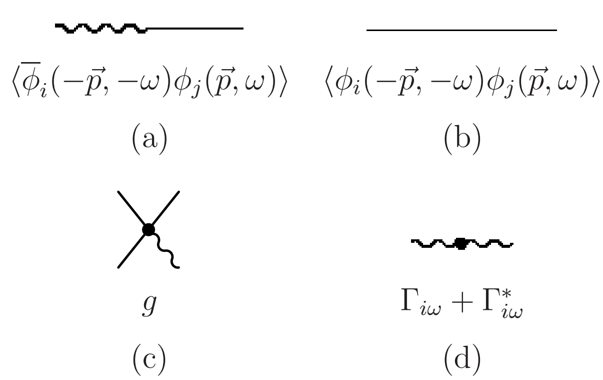


Figure 2. Diagrammatic elements of the perturbation theory: (a) response propagator, (b) correlation propagator, (c) interaction vertex and (d) noise vertex. The straight parts of each line are associated to fields ϕ, whereas wiggled lines correspond to $\overline{\phi}$ fields.

P (with $\gamma_{\mathrm{w}}=0$) renormalization indeed generates a supplementary white noise vertex $\gamma_{\mathrm{w}}\neq 0$ and, on the other hand, such a vertex becomes relevant at a D- and N-dependent value $\alpha_c(D,N)$, where $\alpha_c(D,N)$ shows corrections to the Gaussian cross-over occurring at $\alpha_c=1$ for $D<4$.

The first correction to the noise vertex $\mathcal{V}_2^{0,2}$ is given by the second-order diagram depicted in Fig. 3 which can be conveniently written as the Fourier transform of its

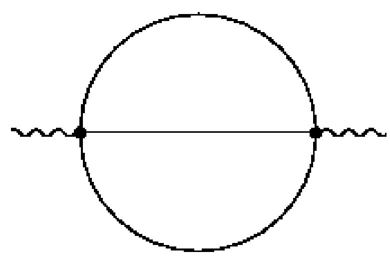


Figure 3. Lowest-order perturbative contribution to the noise vertex.

expression in the time and space domain

$$\begin{aligned}\mathcal{V}_2^{0,2}(\vec{q},\sigma) &= -\frac{g^2(N+2)}{18}\int \mathrm{d}^D x\int_{-\infty}^{+\infty}\mathrm{d}t\ \mathrm{e}^{i\vec{q}\cdot\vec{x}-i\sigma t}\ C_0^3(\vec{x},t)\\ &= -\frac{g^2(N+2)}{9}\int_0^{\infty}\mathrm{d}t\ \cos(\sigma t)\int \mathrm{d}^D x\ \mathrm{e}^{i\vec{q}\cdot\vec{x}}\ C_0^3(\vec{x},t),\end{aligned} \tag{3.20}$$

where the N-dependent prefactor accounts for the combinatorics of the graph (see, e.g., [37]) and C_0 is the Gaussian correlation function with $\gamma_{\mathrm{w}}=0$. In the last line of this equation we used the symmetry $C(\vec{x},t)=C(\vec{x},-t)$. Since we are interested in the critical dynamics, we set r to its critical value $r_c=\mathcal{O}(g)$ (determined, e.g., by the value of r at which $C(\vec{p}=\vec{0},t=0)$ diverges [37, 40, 41]). However, at the order g^2 we are presently interested in, one can neglect the shift of the critical point and

set $r = 0$. The leading behavior of the noise vertex is completely determined by the small-q and small-σ asymptotics of $\mathcal{V}_2^{0,2}$. We can set $q = 0$ from the outset, while the small-frequency limit $\sigma \to 0$ has to be considered with care since the tree-level noise vertex $\mathcal{V}_0^{0,2}(\sigma) = 2\mathrm{Re}\Gamma_{i\sigma}(\gamma_{\mathrm{w}} = 0) = 2\gamma|\sigma|^{\alpha-1}\sin(\pi\alpha/2)$ [see Eq. (3.3)] diverges in this limit for $\alpha < 1$. At the end we shall see that no contribution to $\mathcal{V}_2^{0,2}$ proportional to $|\sigma|^{\alpha-1}$ is actually generated. In what follows we only take the limit $\sigma \to 0$ when it becomes manifestly possible. In this formulation, divergencies arise due to the singular behavior of C_0 at small distances and times, $|\vec{x}| \to 0$ and $t \to 0$. In order to regulize the theory, we introduce a short-distance cut-off ℓ, below which the description in the continuum is no longer considered to be realistic. For example, in lattice models, the cut-off ℓ is naturally identified with the lattice spacing. Analogously, a short-time cut-off is introduced in the convenient form ℓ^z, which is motivated by the scaling behavior discussed above. By using, c.f., the scaling form (B.5) of the Gaussian correlation C_0 influenced by the colored noise (see App. B), the second-order contribution to the regularized vertex function takes the form

$$\mathcal{V}_2^{0,2}(\vec{0},\sigma;\ell) = -\frac{g^2 A_D(N+2)}{9}\int_{\ell^z}^{\infty} \mathrm{d}t\,\cos(\sigma t)\int_{\ell}^{\infty} \mathrm{d}x\, x^{5-2D} g_{C_0}^3(\gamma x^2/t^\alpha), \tag{3.21}$$

where $A_D = 2\pi^{D/2}/\Gamma_E(D/2)$ is the solid angle in D dimensions.

The Wilsonian renormalization scheme (see, e.g., [40]) amounts to a resummation of the perturbation series which is performed according to the following steps:

(I) Effective vertex functions for the 'slow' fluctuations are determined by performing an integration (averaging) over 'fast' fluctuations, within a spatial shell between ℓ and $b\ell$ and at a temporal scale between ℓ^z and $(b\ell)^z$. As a result of this integration the effective vertex functions — and therefore the coupling constants which characterize them — acquire a dependance on the scaling parameter $b > 1$. To be more specific, consider the typical integral which arises in loop calculations and which can be written in the generic form

$$\mathcal{I}(\ell) = \int_{\ell^z}^{\infty} \mathrm{d}t \int_{\ell}^{\infty} \mathrm{d}^D x\; \mathcal{F}(\vec{x},t),$$

with some integrand $\mathcal{F}(\vec{x},t)$. The contribution of the integration over the fast fluctuations is then equivalent to $\mathcal{I}(\ell) - \mathcal{I}(b\ell)$, an expression which we shall use repeatedly below. In the limit $b \to 1$ with $b > 1$ one has $\mathcal{I}(\ell) - \mathcal{I}(b\ell) = -\left[\partial\mathcal{I}(\ell)/\partial\ln\ell\right]\ln b + \mathcal{O}(\ln^2 b)$.

(II) The effective vertex functions calculated in step (I) depend on a new cut-off $b\ell$. In order to recover the original cut-off ℓ one rescales the coordinates and fields in the frequency and momentum domain according to

$$\begin{aligned}
\vec{q} &\mapsto b^{-1}\vec{q},\\
\sigma &\mapsto b^{-z}\sigma,\\
\phi_i &\mapsto b^{D/2+1+z-\eta/2}\phi_i,\\
\overline{\phi}_i &\mapsto b^{D/2+1+z-z_0-\overline{\eta}/2}\overline{\phi}_i.
\end{aligned} \tag{3.22}$$

The resulting expression is multiplied by b^{-D-z} which accounts for the rescaling of the integration measure in the Hamiltonian.

(III) In order to study the evolution of the coupling constants under the renormalization procedure it is convenient to consider the case $b \to 1^+$ which corresponds to an infinitely thin integration shell. In this case the evolution equations for the coupling constants are coupled differential equations that depend upon α and the anomalous dimensions introduced by the rescaling in step (II). The anomalous dimensions are determined by requiring that all coupling constants have a finite asymptotic value under the RG transformation for $b \to \infty$.

Applying step (I) to the noise vertex function $\mathcal{V}_2^{0,2}$ we derive Eq. (3.21) with respect to $\ln \ell$ and we multiply the result by $\ln b$. By defining [see Eq. (C.1)]

$$u^2 \mathcal{E}^{0,2}(\sigma;\gamma) = \frac{\partial \mathcal{V}_2^{0,2}(\vec{0},\sigma;\ell)}{\partial \ln \ell} \tag{3.23}$$

with

$$u = A_D g/(2\pi)^D, \tag{3.24}$$

we find that the effective noise vertex $\mathcal{V}^{0,2}(\vec{0},\sigma \to 0; b\ell)$ for the slow fluctuations with short-time and -distance cut-offs $b\ell$ and $(b\ell)^z$, respectively, is given by

$$\begin{aligned} \mathcal{V}^{0,2}(\vec{0},\sigma \to 0; b\ell) = & -(\Gamma_{i\sigma\to 0} + \Gamma^*_{i\sigma\to 0}) - u^2 \mathcal{E}^{0,2}(0;\gamma)\ln b \\ & + \mathcal{O}(u^2 \ln^2 b, u^3). \end{aligned} \tag{3.25}$$

For details on the calculation of $\mathcal{E}^{0,2}(\sigma;\gamma)$ we refer to App. C.

Clearly, the form of the effective noise vertex has changed, as the term $\mathcal{E}^{0,2}(0;\gamma)$ generated by the non-Gaussian fluctuations has the form of a white-noise contribution, whereas the coefficient γ of the colored noise is not modified up to this order in perturbation theory. As a result, it is convenient to account for the contribution of a white-noise vertex from the outset, by replacing $\Gamma_{i\sigma}$ by $\Gamma_{i\sigma} + \gamma_{\mathrm{w}}$. This implies that the Gaussian correlation function C_0 that determines the loop correction still has a scaling form but with a scaling function g_{C_0} that is now a function of two variables, see Eq. (B.12). The correction $\mathcal{E}^{0,2}$ that is generated depends on both γ and γ_{w}, we denote it by $\mathcal{E}^{0,2}(0;\gamma,\gamma_{\mathrm{w}})$ and we explicitly calculate it in Eq. (C.1).

The effective noise vertex depends on the cut-off $b\ell$. Following step (II) of the renormalization procedure we rescale the effective noise vertex as specified in Eq. (3.22). The coupling strengths of the colored and the white noise γ and γ_{w} become running coupling constants $\gamma(b)$ and $\gamma_{\mathrm{w}}(b)$ and in the limit $b \to 1$ they satisfy the set of coupled differential equations

$$\frac{\partial \gamma}{\partial \ln b} = \left[2 - \alpha z_0 - \frac{\alpha}{2}(\overline{\eta} - \eta) - \eta\right]\gamma + \mathcal{O}(\epsilon^3) \tag{3.26}$$

and

$$\frac{\partial \gamma_{\mathrm{w}}}{\partial \ln b} = \left[2 - z_0 - \frac{\overline{\eta} + \eta}{2}\right]\gamma_{\mathrm{w}} + \frac{z}{2} u^{*2} \mathcal{E}^{0,2}(0;\gamma,\gamma_{\mathrm{w}}) + \mathcal{O}(\epsilon^3), \tag{3.27}$$

valid at the critical point. $u^* = \mathcal{O}(\epsilon)$ is the fixed point value of the coupling constant, i.e., the value at which the effective coupling constant $u(b)$ — obtained by applying the procedure outlined here to the 4-point function — flows for $b \to \infty$ and $D < 4$. For $D > 4$, $u^* = 0$ and the scenario within the Gaussian approximation presented in Sec. 3.1 is not altered by the interaction. Accordingly we focus below on the case $D < 4$. Two additional differential equations can be written by considering how the coupling constant u in $\mathcal{V}^{1,3}$ and the coefficient of the term $\propto q^2$ in $\mathcal{V}^{1,1}(\vec{q}, \ldots)$ are modified by the non-Gaussian fluctuations. In particular, the requirement of an effective b-independent coefficient of q^2 fixes η to its well-known static value [37] (see Sec. 3.3.3 for further details).

In order to determine the critical exponents we demand that the amplitude of the noise vertex in the effective Hamiltonian be constant as explained in step (III) of the renormalization procedure. Neglecting for a while the contribution of the non-Gaussian fluctuations to Eq. (3.26) and Eq. (3.27) (which amounts to setting u^* and the anomalous dimensions to zero), one can easily solve them and recover the Gaussian picture which we anticipated in Sec. 3.1. Indeed, $\gamma(b) \sim b^{2-\alpha z_0}$ whereas $\gamma_{\rm w}(b) \sim b^{2-z_0}$ as $b \to \infty$, which implies $\gamma(b)/\gamma_{\rm w}(b) \sim b^{(1-\alpha)z_0}$. Independently of the value of $z_0 > 0$, this ratio tends to zero for $\alpha > 1$. The associated fixed point is characterized by a finite $\gamma_{\rm w}(b \to \infty)$ with a vanishing $\gamma(b \to \infty)$, i.e., the fixed point $P_{\rm w}$ introduced in Sec. 3.1. In order for $\gamma_{\rm w}(b)$ to stay finite, it is necessary to have $z_0 = z_0^{(\rm w)} = 2$ in Eq. (3.27), as expected from our previous discussion. On the contrary, for $\alpha < 1$, $\gamma(b)/\gamma_{\rm w}(b) \to \infty$ for $b \to \infty$ and the associated fixed point has a finite $\gamma(b \to \infty)$ and a vanishing $\gamma_{\rm w}(b \to \infty)$, corresponding to the fixed point P of Sec. 3.1. The former condition requires $z_0 = z_0^{(\rm col)} = 2/\alpha$ in Eq. (3.26), consistently with the discussion therein.

Including now the effects of non-Gaussian fluctuations, the colored-noise fixed point P with $z_0 = z_0^{(\rm col)} = 2/\alpha$ and $\gamma(b \to \infty) \neq 0$ is characterized by a value of $\overline{\eta}$ such that the lhs of Eq. (3.26) vanishes. This yields

$$\overline{\eta} = \overline{\eta}^{(\rm col)} \equiv \left(1 - \frac{2}{\alpha}\right)\eta + \mathcal{O}(\epsilon^3) = \left(1 - \frac{2}{\alpha}\right)\frac{N+2}{2(N+8)^2}\epsilon^2 + \mathcal{O}(\epsilon^3)\,. \quad (3.28)$$

We replaced η by its static value given in [37, 40] since, as we shall show in Sec. 3.3.3, it is α-independent. The value $z^{(\rm col)}$ of z at this fixed point is determined via Eq. (3.18)

$$z = z^{(\rm col)} = \frac{2-\eta}{\alpha} + \mathcal{O}(\epsilon^3) = \frac{2}{\alpha}\left[1 - \frac{N+2}{4(N+8)^2}\epsilon^2\right] + \mathcal{O}(\epsilon^3). \quad (3.29)$$

The fixed point P is stable in the (α, D)-plane (region C in Fig. 4) as long as the value of $\gamma_{\rm w}(b)$ determined by Eq. (3.27) at the fixed-point P with $\overline{\eta} = \overline{\eta}^{(\rm col)}$ and $z_0 = z_0^{(\rm col)} = 2/\alpha$ stays finite for $b \to \infty$. A crossover towards the fixed-point $P_{\rm w}$ in the (α, D)-plane (region W in Fig. 4) controlled by the white noise occurs as soon as $\gamma_{\rm w}(b \to \infty) \to \infty$. In this limit, $\mathcal{E}^{0,2}(0; \gamma, \gamma_{\rm w} \to \infty) \simeq \gamma_{\rm w}\mathcal{E}^{0,2}_{\rm w}$ independently of γ [as long as $\gamma(b)$ remains finite, see Eq. (C.5) for details] where the constant $\mathcal{E}^{0,2}_{\rm w}$ is given in Eq. (C.6) and is such that

$$u^{*2}\mathcal{E}^{0,2}_{\rm w} = \frac{N+2}{(N+8)^2} 3 \ln\frac{4}{3}\epsilon^2 + \mathcal{O}(\epsilon^3)\,. \quad (3.30)$$

Thus, the equation which determines the evolution of $\gamma_{\rm w}$ at the fixed point P becomes

$$\frac{\partial\gamma_{\rm w}}{\partial\ln b} = \left[2 - z_0^{(\rm col)} - \frac{\overline{\eta}^{(\rm col)}+\eta}{2} + \frac{z^{(\rm col)}}{2}u^{*2}\mathcal{E}_{\rm w}^{0,2}\right]\gamma_{\rm w} + \mathcal{O}(u^{*3}), \tag{3.31}$$

and the crossover occurs as soon as the the quantity in brackets changes sign. The expression of the crossover line is readily determined by taking into account the values of $z_0^{(\rm col)}$, $\overline{\eta}^{(\rm col)}$, $z^{(\rm col)}$, and $\mathcal{E}_{\rm w}^{0,2}$ reported in Eqs. (3.28), (3.29), and (3.30):

$$\alpha_c = 1 - \frac{3}{2}\ln\frac{4}{3}\,\frac{N+2}{(N+8)^2}\,\epsilon^2 + \mathcal{O}(\epsilon^3). \tag{3.32}$$

For $\alpha > \alpha_c$ (region W in Fig. 4), $\gamma_{\rm w}(b\to\infty)\to\infty$ and the point P is no longer a fixed point of the rescaled effective action, as the white-noise contribution becomes predominant. In order for it to become constant and therefore to determine the fixed point $P_{\rm w}$, $\overline{\eta}$ in Eq. (3.27) should now take the value $\overline{\eta}^{(\rm w)}$ such that $\partial\gamma_{\rm w}/\partial\ln b = 0$, with $z_0 = z_0^{(\rm w)} = 2$. Assuming that the coefficient $\gamma(b)$ of the colored noise vanishes asymptotically for $b\to\infty$, the lhs of Eq. (3.27) becomes $-(\overline{\eta}^{(\rm w)}+\eta)/2 + (z/2)u^{*2}\mathcal{E}_{\rm w}^{0,2}$, where we used the fact that $\mathcal{E}^{0,2}(0;\gamma=0,\gamma_{\rm w}) = \gamma_{\rm w}\mathcal{E}_{\rm w}^{0,2}$. The condition that the rhs of the same equation vanishes implies

$$\overline{\eta} = \overline{\eta}^{(\rm w)} = -\eta + 2u^{*2}\mathcal{E}_{\rm w}^{0,2} + \mathcal{O}(\epsilon^3) = \frac{N+2}{(N+8)^2}\left[6\ln\frac{4}{3} - \frac{1}{2}\right]\epsilon^2 + \mathcal{O}(\epsilon^3) \tag{3.33}$$

and from Eq. (3.18),

$$z = z^{(\rm w)} = 2 + \frac{N+2}{(N+8)^2}\left[3\ln\frac{4}{3} - \frac{1}{2}\right]\epsilon^2 + \mathcal{O}(\epsilon^3) \tag{3.34}$$

in agreement with [3, 37]. In order to verify the consistency of the assumption $\gamma(b)\to 0$ for $b\to\infty$ under which Eq. (3.33) has been derived, one can specialize Eq. (3.26) to the white-noise fixed point $P_{\rm w}$, by using $\overline{\eta}^{(\rm w)}$, $z_0^{(\rm w)} = 2$, and $z^{(\rm w)}$ [see Eq. (3.34)] as the values of $\overline{\eta}$, z_0, and z. Accordingly, the term in parenthesis in the rhs can be written as $-2(\alpha-\alpha_c)+\mathcal{O}(\epsilon^3)$ and therefore $\gamma(b)\sim b^{-2(\alpha-\alpha_c)}$ indeed vanishes for $\alpha>\alpha_c$ as $b\to\infty$. This also proves that the white-noise fixed point $P_{\rm w}$ is stable against the perturbation of the colored noise as long as $\alpha>\alpha_c$, a statement which complements the one presented above about the stability of P.

Summarizing, Eq. (3.32) determines the line in the (α, D)-plane which separates region W from region C: in the former, the white noise dominates and $z = z^{(\rm w)}$ (in agreement with [3, 37]), whereas in the latter the colored noise dominates and $z = z^{(\rm col)}$ is given by Eq. (3.29). Figure 4 illustrates this scenario for $N = 1, 4, \infty$.

3.3.2. Renormalization of the self-energy and FDT. The fluctuation-dissipation theorem (FDT) expressed in Eq. (2.12) is a consequence of a symmetry of the action in equilibrium [10, 35] and it has to be preserved under renormalization. Therefore, the noise vertex and the memory kernel have to be related by the FDT even beyond the Gaussian approximation, that we analyzed in Sec. 3.1. Here we explicitly show that this relation is still valid when non-Gaussian corrections up to $\mathcal{O}(\epsilon^2)$ (or, equivalently,

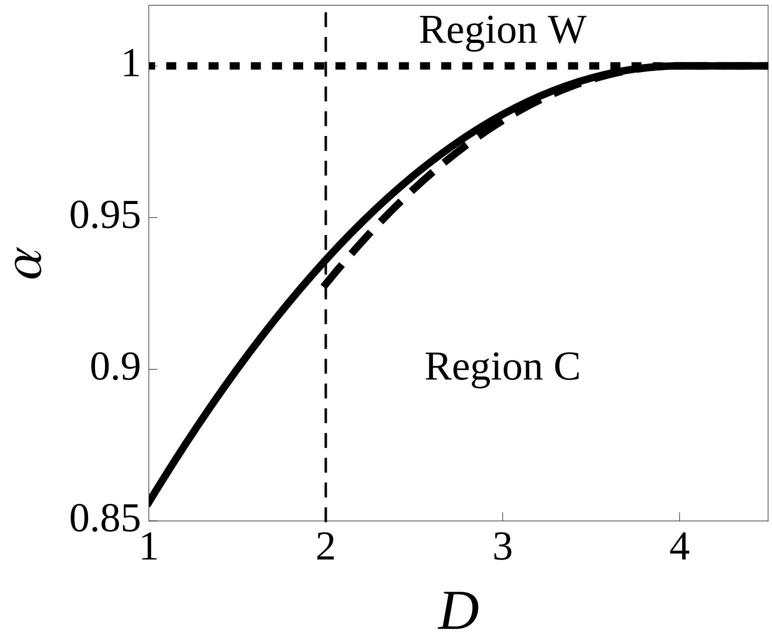


Figure 4. Boundary between the regions W and C of the (α, D)-plane characterized, respectively, by white and colored noise. The boundary curve $\alpha = \alpha_c(D, N)$ as a function of the spatial dimensionality D is reported here for $N = 1$ (solid line, Ising universality class), 4 (dashed), and ∞ (dotted) where the $\mathcal{O}((4-D)^3)$-correction in the corresponding perturbative expression (3.32) for $D < 4$ has been neglected. The vertical dashed line indicates the lower critical dimensionality of the model for $N > 1$. The coefficient of the term $\mathcal{O}((4-D)^2)$ in Eq. (3.32) takes its maximum value for $N = 4$ (dashed curve) and then it decreases monotonically as a function of N, vanishing for $N \to \infty$. For $D > 4$, α_c takes the D-independent Gaussian value $\alpha_{c,0} = 1$ (dotted line). Clearly, the dependence of the boundary curve on the dimensionality D is quantitatively rather weak.

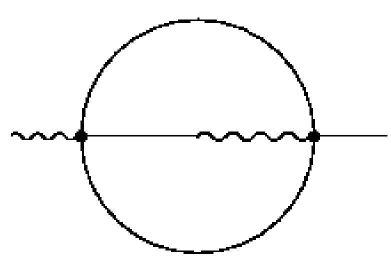


Figure 5. Second-order contribution to the self-energy.

$\mathcal{O}(g^2)$) are accounted for. The first correction to the memory kernel comes from the second-order self-energy contribution

$$\mathcal{V}_2^{1,1}(\vec{q}, \sigma) = -\frac{g^2(N+2)}{6} \int_0^\infty \mathrm{d}t \int \mathrm{d}^D x \; \mathrm{e}^{i\vec{q}\cdot\vec{x} - i\sigma t} \, C_0^2(\vec{x}, t) R_0(\vec{x}, t) \tag{3.35}$$

represented in Fig. 5. Note that R_0 is causal and restricts the time integral to run over positive values only. The expansion of this expression as a power series in σ and $\vec{q}$ allows one to identify the terms which contribute to the renormalization of the different

parameters of the Gaussian vertex $\mathcal{V}_0^{1,1}(\vec{q},\sigma) = i\sigma\Gamma_{i\sigma} + q^2 + r$. The terms which are independent of both σ and $\vec{q}$ contribute to the renormalization of the parameter r (which is also modified by an $\mathcal{O}(g)$ term not discussed here), the terms proportional to $\sigma^0 q^2$ contribute to the renormalization of the fields and those proportional to $i\sigma^1 q^0$ to the renormalization of the memory kernel Γ. First of all we observe that the FDT given in Eq. (2.12) allows us to express R_0 in Eq. (3.35) as $-\partial_t C_0$. An integration by parts yields

$$\mathcal{V}_2^{1,1}(\vec{q},\sigma) = -\frac{g^2(N+2)}{18}\int \mathrm{d}^D x\ \mathrm{e}^{i\vec{q}\cdot\vec{x}}\left\{C_0^3(\vec{x},0) - i\sigma\int_0^\infty \mathrm{d}t\ \mathrm{e}^{-i\sigma t}\, C_0^3(\vec{x},t)\right\}. \tag{3.36}$$

Hence,

$$\begin{aligned}\mathrm{Im}\,\mathcal{V}_2^{1,1}(\vec{q},\sigma) &= \sigma\frac{g^2(N+2)}{18}\ \mathrm{Re}\int \mathrm{d}^D x\ \mathrm{e}^{i\vec{q}\cdot\vec{x}}\int_0^\infty \mathrm{d}t\ \mathrm{e}^{-i\sigma t}\, C_0^3(\vec{x},t)\\ &= -\frac{\sigma}{2}\mathcal{V}_2^{0,2}(\vec{q},\sigma),\end{aligned} \tag{3.37}$$

where the last equality follows from a comparison with Eq. (3.20) and shows that the FDT in the frequency domain [see Eq. (3.6)] is satisfied by the corrections $\mathcal{O}(g^2)$. Note that the vertex $\mathcal{V}^{1,1}$ receives also a correction $\mathcal{V}_1^{1,1}$ of $\mathcal{O}(g)$ given by a tadpole diagram which, however, is a real constant and does not contribute to the imaginary part. We conclude that up to and including the second order in the coupling constant $2\,\mathrm{Im}\,\mathcal{V}^{1,1}(\vec{0},\sigma) = -\sigma\mathcal{V}^{0,2}(\vec{0},\sigma)$. (This proof can be readily extended to the corresponding regularized vertex functions, characterized by short- time and distance cut-offs.)

3.3.3. Renormalization of the self-energy: the anomalous exponent η. In the same spirit as before we can deduce the first correction to the static exponent η. It is instructive to see why the dependence upon α does not affect the final result, even though the regularized expression of Eq. (3.35) does via C_0 and R_0. In order to single out the contribution of $\mathcal{V}_2^{1,1}$ to the coefficient of q^2, one expands Eq. (3.35) — suitably regularized as discussed above — up to second order in $\vec{q}$, finding

$$\mathcal{V}_2^{1,1}(\vec{q},\sigma=0;\ell) = \frac{1}{2}q^2 g^2\frac{N+2}{6}\frac{A_D}{D}\int_{\ell^z}^\infty \mathrm{d}t\int_\ell^\infty \mathrm{d}x\ x^{D+1}\ C_0^2(\vec{x},t)R_0(\vec{x},t) + \ldots, \tag{3.38}$$

where the dots indicate all the terms which do not contribute to the field renormalization, i.e., which are not proportional to $\sigma^0 q^2$. In Eq. (3.38) we used the fact that, for a generic function f, $\int \mathrm{d}^D x\, x_i x_j\, f(|\vec{x}|) = (\delta_{ij}/D)\int \mathrm{d}^D x\, |\vec{x}|^2 f(|\vec{x}|)$, which is valid also for the regularized integral. As in the case of Eq. (3.36) one can take advantage of the FDT, Eq. (2.12), to express the integrand in Eq. (3.38) as a total derivative, which can be integrated to yield

$$\mathcal{V}_2^{1,1}(\vec{q},0;\ell) = q^2 g^2\ \frac{A_D(N+2)}{36D}\int_\ell^\infty \mathrm{d}x\ x^{D+1}\ C_0^3(\vec{x},\ell^z) + \ldots \tag{3.39}$$

We note here that even though the (full) dynamic correlation function C (and therefore its Gaussian approximation C_0) depends on the value of α, the static correlation function

$C(\vec{x}, t = 0)$ does not. This is explicitly shown for C_0 in Eq. (B.4). While the limit $\ell \to 0$ of the rhs of Eq. (3.39) cannot be explicitly taken due to the short-distance singularity of the integrand, such a limit can be taken for the correlation function, i.e., $C_0(\vec{x}, t = \ell^z) \simeq C_0(\vec{x}, t = 0)$ and therefore the expression of $\mathcal{V}_2^{1,1}(\vec{q}, 0; \ell \to 0)$ becomes — as expected — independent of α at the leading relevant order in ℓ. Applying the same renormalization procedure as in Sec. 3.3.1, Eq. (3.39) can be used to calculate the effective vertex $\mathcal{V}_2^{1,1}(\vec{q}, 0; b\ell)$ after having integrated out the fast fluctuations. Similarly to Eq. (3.23) one defines

$$q^2 u^2 \mathcal{E}^{1,1} + \ldots = -\frac{\partial \mathcal{V}_2^{1,1}(\vec{q}, 0; \ell)}{\partial \ln \ell}, \tag{3.40}$$

For $b \to 1$, the resulting effective vertex is

$$\mathcal{V}_2^{1,1}(\vec{q}, 0; b\ell) = q^2 + q^2 u^2 \mathcal{E}^{1,1} \ln b + \mathcal{O}(u^2 \ln^2 b, u^3) + \ldots \tag{3.41}$$

In order to recover the original cut-off ℓ we rescale the fields and coupling constants according to Eq. (3.22) and we take $b \to 1$. The part of $\mathcal{V}_2^{1,1}(\vec{q}, 0; \ell)$ that is proportional to $\sigma^0 q^2$ satisfies the evolution equation

$$\frac{\partial \mathcal{V}_2^{1,1}}{\partial \ln b} = -q^2 \left[\eta - u^{*2} \mathcal{E}^{1,1}\right] + \mathcal{O}(u^2 \ln b, u^3) + \ldots. \tag{3.42}$$

By demanding that the amplitude of $\mathcal{V}_2^{1,1}$ be constant and by using the numerical value of $\mathcal{E}^{1,1}$ calculated in Eq. (C.8) we find

$$\eta = u^{*2} \mathcal{E}^{1,1} = \frac{N+2}{2(N+8)^2} \epsilon^2 + \mathcal{O}(\epsilon^3), \tag{3.43}$$

i.e., η has the same α-independent value as in the static theory confirming our expectations alluded to at the beginning of Sec. 3.

4. Non-equilibrium dynamics

4.1. Preliminary remarks

In this Section we investigate the non-equilibrium dynamics assuming that the model is prepared in some initial condition at time $t = 0$ and that it is let relax subsequently at its critical point. This problem has been studied in detail in the white-noise case [10, 13]. The analysis reveals the emergence of an interesting scaling behavior of two-time quantities, usually referred to as *aging* (see, in this context, [14, 36, 39]). More precisely, the relaxation is studied via the field-theoretical action $\mathcal{S}$ in Eqs. (2.8) and (2.9) with $T = 0$, supplemented by a suitable distribution $\mathcal{P}_{IC}$ for the initial condition at time $t = T = 0$. In particular, a high-temperature disordered state is modeled by a Gaussian distribution with zero mean:

$$\ln \mathcal{P}_{IC} = -\int \mathrm{d}^D x \, \frac{\tau_0}{2} \phi^2(\vec{x}, t = 0) \tag{4.1}$$

The parameter τ_0 sets the inverse width of the initial distribution. Within the Gaussian approximation, the field ϕ has a scaling dimension $d_{\phi,0}$ given by Eq. (3.9) in momentum

and frequency space, i.e., a dimension $d_{\phi,0} - z_0 - D$ in the space and time domain. Using this dimension for the initial field $\phi(\vec{x}, t = 0)$ we find that $\tau_0 \to b^2\tau_0$ under rescaling. Consequently, the width of the initial distribution shrinks to zero as $b \to \infty$, leading to a zero effective value of the initial order parameter $\phi(\vec{x}, t = 0) = 0$ and, therefore, to a correlation function with Dirichlet boundary conditions at $t = 0$. In App. D [see Eq. (D.5) and Eq. (B.13)] we show that the Laplace transforms of the Gaussian propagators are:

$$R_0(\vec{p}; \lambda, \kappa) = \frac{1}{(\lambda + \kappa)(\lambda\Gamma_\lambda + p^2 + r)}, \tag{4.2}$$

$$C_0(\vec{p}; \lambda, \kappa) = \frac{\Gamma_\lambda + \Gamma_\kappa}{(\lambda + \kappa)(\lambda\Gamma_\lambda + p^2 + r)(\kappa\Gamma_\kappa + p^2 + r)}, \tag{4.3}$$

where the Laplace transformed noise is $\Gamma_\lambda = \gamma\lambda^{\alpha-1} + \gamma_{\rm w}$ with $\lambda \in \mathcal{R}_+$. [In order to transform Eq. (2.6) for $\alpha > 1$ one should introduce a short-time cut-off. However, as we pointed out after Eq. (3.3), this modification is not necessary as long as one is interested in the leading long-time, near critical dynamic behavior of the system. Accordingly, we shall use this form for Γ_λ irrespective of the value of α.] As in the equilibrium case, the propagators have a simple analytic form in the time domain only for $\alpha = 1$ or $\gamma_{\rm w} = 0$. It is easy to show that the response propagator is the same in and out of equilibrium and, therefore, that it is time-translationally invariant.

The correlation function C_0 can always be written as the sum of the Gaussian equilibrium correlation $C_0^{(\rm e)}$ and the remaining non-equilibrium contribution, which we denote by $C_0^{(\rm ne)}$ and which will play an important role in fixing the genuinely non-equilibrium properties of the relaxation, e.g., the non-equilibrium exponent θ and the effective temperature. The Laplace transform $C_0^{(\rm e)}(\vec{p}, \lambda)$ of the equilibrium correlation function $C_0^{(\rm e)}(\vec{p}, t)$ can be obtained from Eq. (B.21):

$$C_0^{(\rm e)}(\vec{p}, \lambda) = \frac{1}{p^2 + r}\frac{\Gamma_\lambda}{\lambda\Gamma_\lambda + p^2 + r}. \tag{4.4}$$

The full non-equilibrium correlator Eq. (4.3) can be expressed as

$$C_0(\vec{p}, \lambda, \kappa) = \frac{C_0^{(\rm e)}(\vec{p}, \lambda) + C_0^{(\rm e)}(\vec{p}, \kappa)}{\lambda + \kappa} - (p^2 + r)C_0^{(\rm e)}(\vec{p}, \lambda)C_0^{(\rm e)}(\vec{p}, \kappa) \tag{4.5}$$

which displays the fact that C_0 is the sum of an equilibrium time-translationally invariant term and a non-stationary term. Indeed, the Laplace transform $\mathcal{L}[F](\lambda, \kappa)$ with respect to both t and t' of any (translationally invariant) function $F(|t - t'|)$ is given by $\mathcal{L}[F](\lambda, \kappa) = (F_\lambda + F_\kappa)/(\lambda + \kappa)$, which is exactly the form of the first term in Eq. (4.5) [28]. Accordingly, we can identify the non-equilibrium part $C_0^{(\rm ne)}$ of C_0 as $C_0^{(\rm ne)}(\vec{p}, \lambda, \kappa) \equiv -(p^2 + r)C_0^{(\rm e)}(\vec{p}, \lambda)C_0^{(\rm e)}(\vec{p}, \kappa)$ which translates, by virtue of Eq. (B.21), into the non-stationary expression $(t, t' > 0)$

$$C_0^{(\rm ne)}(\vec{p}; t, t') = -\frac{1}{p^2 + r}E_\alpha(-(p^2 + r)t^\alpha/\gamma)E_\alpha(-(p^2 + r)t'^\alpha/\gamma), \tag{4.6}$$

where we wrote the equilibrium Gaussian correlation function in terms of the Mittag-Leffler function, anticipated in Eq. (3.8) and discussed in App. B.2.

4.2. General non-equilibrium renormalization group analysis

The addition of the initial condition to the action modifies the scaling of the fields at the boundary $t = 0$ compared to the one in the 'time bulk' $t > 0$ [10]. In addition, to the bulk renormalization a new initial time renormalization is required, which gives rise to contributions 'located' at the time surface (that is the hyperplane determined by the condition $t = 0$). These can be absorbed by introducing a new anomalous dimension of the initial field $\overline{\phi}_0(\vec{p}) \equiv \overline{\phi}(\vec{p}, 0)$ (see [42] for an application to surface critical phenomena). The general scaling of the initial fields in the time and momentum domain reads [cf. Eqs. (3.16) and (3.17)]

$$\begin{array}{rcl} \phi(b^{-1}\vec{p}, t = 0) & \mapsto & b^{D/2+1-\eta/2}\, \phi(\vec{p}, 0), \\ \overline{\phi}(b^{-1}\vec{p}, t = 0) & \mapsto & b^{D/2+1-z_0-\overline{\eta}/2-\overline{\eta}_{\rm in}/2}\, \overline{\phi}(\vec{p}, 0), \end{array} \tag{4.7}$$

where $\overline{\eta}_{\rm in}$ is a new exponent, with a Gaussian value $\overline{\eta}_{\rm in,0} = 0$. Note that the anomalous dimension of the initial response field $\overline{\phi}(\vec{p}, 0)$ is allowed to differ by $\overline{\eta}_{\rm in}$ from its bulk value. In [10, 14] one can find a careful analysis for the white-noise case where it is explained why only the initial response field has to be renormalized. Here, we make the same assumption and we check its validity *a posteriori*. $\overline{\eta}_{\rm in}$ is related to the so-called initial-slip exponent θ [introduced at the end of Sec. 2, see Eqs. (2.13) and (2.14)] by [10]

$$\theta = -\overline{\eta}_{\rm in}/(2z). \tag{4.8}$$

The analysis in [10, 14] has to be slightly modified to deal with colored noise. Our starting point is the general leading scaling behavior of the critical correlation functions $\mathcal{G}^{n,\overline{n},\overline{n}_0}$ of n bulk fields ϕ, $\overline{n}$ bulk response fields $\overline{\phi}$ and $\overline{n}_0$ initial response fields $\overline{\phi}_0$, evaluated at the set of points $\{\vec{p}, t\}$ in momentum and time:

$$\mathcal{G}^{n,\overline{n},\overline{n}_0}(\{\vec{p}, t\}) \simeq b^{-\delta(n,\overline{n},\overline{n}_0)} \mathcal{G}^{n,\overline{n},\overline{n}_0}(\{b^{-1}\vec{p}, b^z t\}), \tag{4.9}$$

where $\delta(n, \overline{n}, \overline{n}_0) = -D + n(D/2+1-\eta/2) + \overline{n}(D/2+1-z_0-\overline{\eta}/2) + \overline{n}_0(D/2+1-z_0-\overline{\eta}/2-\overline{\eta}_{\rm in}/2)$. [In writing Eq. (4.9) and the analogous relations presented below, we always understand that the correlation functions on the lhs and rhs are characterized by the different length cut-offs $b\ell$ and ℓ, respectively.] This scaling behavior is a consequence of the scaling dimensions of the fields ϕ, $\overline{\phi}$ and $\overline{\phi}_0$ as functions of time and momentum [compare to Eqs. (3.16), (3.17) and (4.7)] and the dimension of the δ-function ensuring the total momentum conservation. Note that all correlation functions with an initial field $\phi_0 \equiv \phi(\vec{p}, 0)$ vanish [see the discussion at the end of Sec. 4.1]. Specifically, the two point correlation and response functions ($\mathcal{G}^{2,0,0}$ and $\mathcal{G}^{1,1,0}$, respectively) scale as

$$C(\vec{p}; t, t') \simeq b^{\eta-2}\, C(\vec{p}/b; b^z t, b^z t'), \tag{4.10}$$

$$R(\vec{p}; t, t') \simeq b^{z_0-2+\eta/2+\overline{\eta}/2}\, R(\vec{p}/b; b^z t, b^z t'). \tag{4.11}$$

By choosing $b = (t - t')^{-1/z}$ these scaling forms become

$$\begin{array}{rcl} C(\vec{p}; t, t') & \simeq & (t - t')^{(2-\eta)/z}\, \tilde{F}_C((t - t')^{1/z}\vec{p}, t'/t), \\ R(\vec{p}; t, t') & \simeq & (t - t')^{(2-z-\eta)/z}\, \tilde{F}_R((t - t')^{1/z}\vec{p}, t'/t). \end{array} \tag{4.12}$$

In general the scaling functions $\tilde{F}_C$ and $\tilde{F}_R$ are not expected to have a finite, non-vanishing value for $t' \to 0$. In order to deduce their behavior for small t' we employ a

short-distance expansion [37] of the fields $\phi(\vec{p},t')$ and $\overline{\phi}(\vec{p},t')$ around $t'=0$. However, these are not independent. Indeed, the full correlation and linear response functions verify the equations [10]

$$\begin{aligned} C(\vec{p};t,t') \;=\; & \int_0^t \mathrm{d}s \int_0^s \mathrm{d}s'\; R_0(\vec{p};t,s)\tilde{\mathcal{V}}^{1,1}(\vec{p},s,s')C_0(\vec{p};t',s') \\ & + \int_0^{t'} \mathrm{d}s' \int_0^{s'} \mathrm{d}s\; C_0(\vec{p};t,s)\tilde{\mathcal{V}}^{1,1}(\vec{p},s',s)R_0(\vec{p};t',s') \\ & + \int_0^t \mathrm{d}s \int_0^{t'} \mathrm{d}s'\; R_0(\vec{p};t,s)\tilde{\mathcal{V}}^{0,2}(\vec{p},s,s')R_0(\vec{p};t',s') \end{aligned} \tag{4.13}$$

and

$$R(\vec{p};t,t') = \int_{t'}^{t} \mathrm{d}s \int_{t'}^{s} \mathrm{d}s'\; R_0(\vec{p};t,s)\tilde{\mathcal{V}}^{1,1}(\vec{p},s,s')R_0(\vec{p};s',t'), \tag{4.14}$$

where $\tilde{\mathcal{V}}^{n,\overline{n}}$ are the (not necessarily one particle irreducible in contrast to the ones introduced in Sec. 3.3) vertex functions with n amputated external field and $\overline{n}$ amputated external response field legs, respectively. In writing these expressions we accounted for the causality of $\tilde{\mathcal{V}}^{1,1}(\vec{p},s,s') \propto \Theta(s-s')$. After taking the derivative of Eq. (4.13) with respect to t' only the first term in the rhs survives in the limit $t' \to 0$ [note that $R_0(\vec{p};s,s)=0$ [35]]. By comparing the resulting expression with the rhs of Eq. (4.14) one notices that the equations differ only by the last factor in their integrands, $\partial_{t'}C_0(\vec{p};t',s')$ and $R_0(\vec{p};s',t')$, respectively. We deduce that if a relation between the dimensions of the time-derivative of the initial field and the initial response field exists within the Gaussian approximation, it should be preserved when non-Gaussian fluctuations are accounted for. Let us then examine the propagators. We focus on region C where they satisfy the equation,

$$\partial_{t'}C_0(\vec{p};t,t'\to 0) \simeq t'^{\alpha-1}\int_0^t \mathrm{d}s\,\Gamma(t-s)R_0(\vec{p};s,t'\to 0) \tag{4.15}$$

[proven in App. D, see Eq. (D.8)]. In the early t' limit we formally expand the fields according to

$$\phi(\vec{p},t'\to 0) \sim \varphi(t')\dot{\phi}_0(\vec{p}) \quad \text{and} \quad \overline{\phi}(\vec{p},t'\to 0) \sim \overline{\varphi}(t')\overline{\phi}_0(\vec{p}). \tag{4.16}$$

ϕ is proportional to $\dot{\phi}_0(p)$ and $\overline{\phi}$ is proportional to $\overline{\phi}_0(p)$ since the former vanishes while the latter is allowed to be finite for $t' \to 0$. We see from Eq. (4.15) that, under the rescaling $t \to b^z t$, $s \to b^z s$ and $p \to p/b$ (leaving t' unchanged), the scaling dimensions $d_{\dot{\phi}_0}$ and $d_{\overline{\phi}_0}$ of $\dot{\phi}_0$ and $\overline{\phi}_0$, respectively, verify

$$d_{\dot{\phi}_0} = z(1-\alpha) + d_{\overline{\phi}_0}. \tag{4.17}$$

For $\alpha = 1$ this reduces to the relation found in [10]. The expansion of $\overline{\phi}$ in Eq. (4.16) can be used to calculate the correlation function $\mathcal{G}^{1,1,0}(\vec{p};t,t'\to 0) \sim \overline{\varphi}(t')\mathcal{G}^{1,0,1}(\vec{p};t)$ and by matching the scaling dimensions of the lhs and rhs with the help of Eq. (4.9) we conclude that $\overline{\varphi}(t') \sim t'^{-\theta}$ where θ is given by Eq. (4.8). Besides, the rescaling of t' (keeping t and s unchanged) implies $\varphi(t') \sim t'^{\alpha-\theta}$ if the scaling dimensions of the lhs and rhs in Eq. (4.15) are to match. Hence, the small-t' limit of the response function is

$$R(\vec{p};t,t'\to 0) \sim \overline{\varphi}(t')\langle\phi(-\vec{p},t)\overline{\phi}_0(\vec{p})\rangle \sim t'^{-\theta}\mathcal{G}^{1,0,1}(\vec{p},t), \tag{4.18}$$

where we introduced a short-hand notation for the arguments of $\mathcal{G}^{1,0,1}$ in which we only write the non-vanishing time t. The scaling properties of $\mathcal{G}^{1,0,1}(\vec{p},t)$ are given by Eq. (4.9):

$$\begin{aligned}\mathcal{G}^{1,0,1}(\vec{p},t) &\simeq t^{-(\eta/2+\overline{\eta}/2+\overline{\eta}_{\rm in}/2+z_0-2)/z}\,\mathcal{G}^{1,0,1}(t^{1/z}\vec{p},1)\\ &= t^{(2-\eta-z)/z+\theta}\,\mathcal{G}^{1,0,1}(t^{1/z}\vec{p},1)\end{aligned} \tag{4.19}$$

where we used the relation between anomalous and dynamic exponents, Eq. (3.18), and the relation between θ and $\overline{\eta}_{\rm in}$, Eq. (4.8). Consequently, taking Eqs. (4.12), (4.17) and (4.19) into account, we conclude that

$$R(\vec{p};t,t'\to 0) \simeq t^{(2-z-\eta)/z}\left(\frac{t}{t'}\right)^{\theta} F_R(t^{1/z}\vec{p},0). \tag{4.20}$$

A similar analysis of the scaling behavior of the correlation, taking into account Eq. (4.17), yields

$$C(\vec{p};t,t'\to 0) \simeq t^{(2-\eta)/z}\left(\frac{t}{t'}\right)^{\theta-\alpha} F_C(t^{1/z}\vec{p},0). \tag{4.21}$$

These results are used to capture the singular behavior of the scaling functions in Eq. (4.12) by writing:

$$R(\vec{p};t,t') \simeq (t-t')^{(2-z-\eta)/z}\left(\frac{t}{t'}\right)^{\theta} F_R((t-t')^{1/z}\vec{p},t'/t), \tag{4.22}$$

$$C(\vec{p};t,t') \simeq (t-t')^{(2-\eta)/z}\left(\frac{t}{t'}\right)^{\theta-\hat{\alpha}} F_C((t-t')^{1/z}\vec{p},t'/t), \tag{4.23}$$

with

$$\hat{\alpha} = \begin{cases} 1 & \text{for } \alpha \geq \alpha_c(D,N),\\ \alpha & \text{for } \alpha < \alpha_c(D,N),\end{cases} \tag{4.24}$$

which encompass the white noise result $\hat{\alpha}=1$ [10] for $\alpha \geq \alpha_c$. The scaling functions F_C and F_R are regular for $t' \to 0$ and depend on α. Moreover, in the RG sense they are universal functions up to an overall amplitude and the normalization of their first argument.

The emergence of $\hat{\alpha} \neq 1$ for colored noise can be checked within the Gaussian approximation by looking at the initial-slip behavior of the propagators R_0 and C_0 with $\alpha < 1$. First of all, note that θ takes the value $\theta_0 = 0$ within the Gaussian theory, as one can infer by comparing the scaling form Eq. (4.22) with the expression for the non-equilibrium response R_0 at criticality, which coincides with the equilibrium one in Eq. (3.7) and is invariant under time translations. Using this value θ_0 of θ one has $\lim\limits_{\kappa\to\infty}\kappa C_0(\vec{p};\lambda,\kappa) \sim \kappa^{-\alpha}$ and $\lim\limits_{\kappa\to\infty}\kappa R_0(\vec{p};\lambda,\kappa) \sim \kappa^{0}$ from Eqs. (4.2) and (4.3), respectively.

4.2.1. The initial-slip exponent θ. Out of equilibrium the first correction to the self energy leads to a modification of the scaling of the initial response field. The

response function up to first order in the perturbative expansion reads, for zero external momentum,

$$R(\vec{0};t,t';\ell) = R_0(\vec{0};t,t') + \int_{t'}^{t} \mathrm{d}s\ R_0(\vec{0};t,s)B_{\ell^{-1}}(s)R_0(\vec{0};s,t'). \tag{4.25}$$

$B_{\ell^{-1}}(s)$ stands for the 'tadpole' diagram represented in Fig. 6, which can be calculated by using standard Feynman rules in the time domain [1, 2, 7], whereas ℓ^{-1} is the large-momentum cut-off introduced in order to regularize the otherwise divergent integral defining $B_{\ell^{-1}}(s)$:

$$B_{\ell^{-1}}(s) = -\frac{g(N+2)}{6}\int_{|\vec{p}|<\ell^{-1}} \frac{\mathrm{d}^D p}{(2\pi)^D}\ C_0(\vec{p};s,s). \tag{4.26}$$

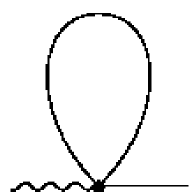


Figure 6. First order contribution to the non-equilibrium self-energy.

The renormalization of the initial response field is due to the non-equilibrium part $C_0^{(\mathrm{ne})}$ of C_0. Indeed, the equilibrium part $C_0^{(\mathrm{e})}$ is characterized by time-translation invariance and therefore it contributes with a time-independent function of $\vec{p}$ to $C_0(\vec{p};s,s)$ in Eq. (4.26). In turn, such a function results in a time-independent contribution $B_{\ell^{-1}}^{(\mathrm{e})}$ to $B_{\ell^{-1}}$, which can be thought of as due to a shift $r \mapsto r - B_{\ell^{-1}}^{(\mathrm{e})}$ of the mass r in the expression of the response functions R_0 appearing in the rhs of Eq. (4.25), i.e., as a mass renormalization. [We recall that $R_0(\vec{p};t,t')$ actually depends on the two times via $t-t'$.] One can check that this term yields the correct first order correction to the critical exponent ν which is the same as in the static theory.

In view of the renormalization procedure outlined in Sec. 3.3.1 we need to calculate

$$\ell^{-1}\partial_{\ell^{-1}}B_{\ell^{-1}}(t) = -\frac{u(N+2)}{6}\ell^{-4}\ C_0^{(\mathrm{ne})}(|\vec{p}| = \ell^{-1};t,t) \tag{4.27}$$

in the limit $\ell \to 0$, for $r=0$ and $D=4$. By using the asymptotic expansion of the generalized Mittag-Leffler functions Eq. (B.19) and their definition, Eq. (B.18), one finds

$$E_\alpha(x) = E_{\alpha,1}(x) = \begin{cases} (-x)^{-1}/\Gamma_E(1-\alpha) & \text{for} \quad x \to -\infty \\ E_\alpha(0) = 1 & \text{for} \quad |x| \ll 1 \end{cases} \tag{4.28}$$

and therefore, using Eq. (4.6),

$$\ell^{-4}C_0^{(\mathrm{ne})}(\ell^{-1};t,t) = -\ell^{-2}E_\alpha^2(-t^\alpha/(\gamma\ell^2)) = \begin{cases} \mathcal{O}(\ell^2/t^{2\alpha}) & \text{for} \quad \ell^{-2}t^\alpha/\gamma \gg 1, \\ \mathcal{O}(\ell^{-2}) & \text{for} \quad \ell^{-2}t^\alpha/\gamma \ll 1. \end{cases} \tag{4.29}$$

Accordingly, $\ell^{-1}\partial_{\ell^{-1}}B_{\ell^{-1}}(t) \to 0$ in the limit $\ell \to 0$ for every fixed $t > 0$. The physical interpretation of this fact is that only the initial field is renormalized by

Eq. (4.25). Indeed the rhs of Eq. (4.29) for finite ℓ provides an approximation of the delta distribution restricted to $z \in \mathbb{R}^+$, usually denoted by $\delta_+(z)$:

$$(\gamma\ell^2)^{-1} E_\alpha^2(-z/(\gamma\ell^2)) \longrightarrow \frac{d(\alpha)}{2}\delta_+(z) \quad \text{for} \quad \ell \to 0, \tag{4.30}$$

where the normalization constant $d(\alpha)$ is given by

$$d(\alpha) = 2\int_0^\infty \mathrm{d}z\, E_\alpha^2(-z), \tag{4.31}$$

and the additional factor 1/2 on the rhs of Eq. (4.30) has been introduced for later convenience in order to have $d(1) = 1$. Taking advantage of the closed-form expressions of the Mittag-Leffler function for $\alpha = 1$, $1/2$, and 0, i.e., $E_1(-z) = \exp(-z)$, $E_{1/2}(-z) = (2/\sqrt{\pi}) \int_z^\infty \mathrm{d}t\, \mathrm{e}^{z^2-t^2}$, and $E_0(-z) = 1/(1+z)$ [43], respectively, it is possible to calculate the corresponding values of the α-dependent constant $d(\alpha)$. One finds $d(1) = 1$, $d(0) = 2$ and, after some algebra, $d(1/2) = \sqrt{2/\pi}\ln(3+2\sqrt{2}) = 1.406\ldots$. Hence,

$$\frac{\partial B_{\ell^{-1}}(t)}{\partial \ln \ell^{-1}} \longrightarrow \frac{u\gamma(N+2)d(\alpha)}{12}\,\delta_+(t^\alpha) \quad \text{for} \quad \ell \to 0. \tag{4.32}$$

Using this expression and the one of the zero-momentum response function $R_0(\vec{0}; t, s) = (t-s)^{\alpha-1}/[\gamma\Gamma_E(\alpha)]$ at criticality which follows from Eq. (B.17), the derivative of the tadpole contribution to the rhs of Eq. (4.25) can be written as

$$\frac{\partial}{\partial \ln \ell^{-1}} \int_s^t \mathrm{d}s\; R_0(\vec{0}; t, s) B_{\ell^{-1}}(s) R_0(\vec{0}; s, t') = \delta_{t',0}\frac{u(N+2)}{12}\frac{d(\alpha)}{\alpha\Gamma_E(\alpha)} R_0(\vec{0}; t, t') \tag{4.33}$$

where $\delta_{t',0} = 1$ for $t' = 0$ and 0 otherwise, illustrating the fact that only the initial field is renormalized. In deriving this last equation we used the fact that $\delta_+(t^\alpha) = \delta_+(t)/(\alpha t^{\alpha-1})$. Altogether, the effective response function with cut-off $b\ell$ reads

$$R(\vec{0}; t, t'; b\ell) = R_0(\vec{0}; t, t')\left[1 + \delta_{t',0}\frac{u(N+2)d(\alpha)}{12\alpha\Gamma_E(\alpha)}\ln b\right]. \tag{4.34}$$

In order to recover the original cut-off ℓ we make use of the scaling relation Eq. (4.9) with $t' = 0$. By taking into account that $\eta = \overline{\eta} = \mathcal{O}(\epsilon^2)$ we have

$$\begin{aligned} R(\vec{0}; t, 0; b\ell) &\simeq b^{-2+z_0+\overline{\eta}_{\mathrm{in}}/2} R(\vec{0}; b^z t, 0; \ell) && (4.35)\\ &= R_0(\vec{0}; t, 0)\; b^{\overline{\eta}_{\mathrm{in}}/2}\left[1 + \frac{u(N+2)d(\alpha)}{12\alpha\Gamma_E(\alpha)}\ln b\right] && (4.36)\\ &= R_0(\vec{0}; t, 0)\left[1 + \frac{\overline{\eta}_{\mathrm{in}}}{2}\ln b + \frac{u(N+2)d(\alpha)}{12\alpha\Gamma_E(\alpha)}\ln b\right]. && (4.37) \end{aligned}$$

By requiring that the amplitude of the response function be constant at the fixed point u^* we obtain

$$\overline{\eta}_{\mathrm{in}} = -\frac{(N+2)d(\alpha)}{(N+8)\alpha\Gamma_E(\alpha)}\epsilon + \mathcal{O}(\epsilon^2) \tag{4.38}$$

whence we find the α-dependent initial slip exponent from Eq. (4.8)

$$\theta = -\frac{\alpha}{4}\overline{\eta}_{\mathrm{in}} = \frac{(N+2)d(\alpha)}{4(N+8)\Gamma_E(\alpha)}\epsilon + \mathcal{O}(\epsilon^2). \tag{4.39}$$

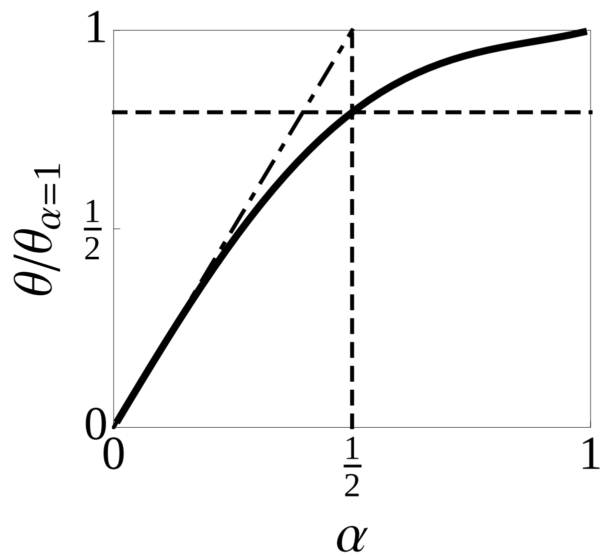


Figure 7. Ratio $\theta/\theta_{\alpha=1}$ between the initial-slip exponent θ in Eq. (4.39) and its white-noise value $\theta_{\alpha=1}$, as a function of α, within the relevant range $0 < \alpha \leq 1$ at the first order in the ϵ-expansion. The dashed horizontal line indicates the value $d(1/2)/\sqrt{\pi} = (\sqrt{2}/\pi)\ln(3+2\sqrt{2}) = 0.793\ldots$ corresponding to $\alpha = 1/2$ (vertical dashed line). The dash-dotted line points out the linear behavior $\theta/\theta_{\alpha=1} \simeq 2\alpha$ expected for $\alpha \to 0$.

In the white noise case $\alpha = 1$ we obtain $\theta = (N+2)\epsilon/[4(N+8)] + \mathcal{O}(\epsilon^2)$ in agreement with the first order result reported in [10]. The dependence of θ on α is shown in Fig. 7. θ increases monotonically from $\theta = 0$ at $\alpha = 0$ to $\theta = \theta(1)$ at $\alpha = 1$ which is the cross-over value up to $\mathcal{O}(\epsilon)$.

4.2.2. Fluctuation-dissipation ratio and effective temperature. A system which equilibrates after a certain finite relaxation time satisfies the FDT. More generally, one defines the *fluctuation-dissipation ratio* (FDR) by

$$X(\vec{p};t,t') = \frac{\beta^{-1}R(\vec{p};t,t')}{\partial_{t'}C(\vec{p};t,t')}, \tag{4.40}$$

where β^{-1} is the temperature of the thermal bath (set to 1 in the previous analysis). In glassy and weakly driven macroscopic systems with slow dynamics — small entropy production limit — this ratio approaches a constant on asymptotic two-time regimes in which, moreover, it is independent of the observable used to define the correlation and associated linear response and admits the interpretation of an effective temperature [30, 31]. For systems with critical points, the asymptotic value

$$X^\infty = \lim_{t'\to\infty}\lim_{t\to\infty} X(\vec{0},t,t'), \tag{4.41}$$

has been suggested to behave as a universal property [32] and, moreover, as an *effective temperature*,

$$\beta^\infty = \beta X^\infty. \tag{4.42}$$

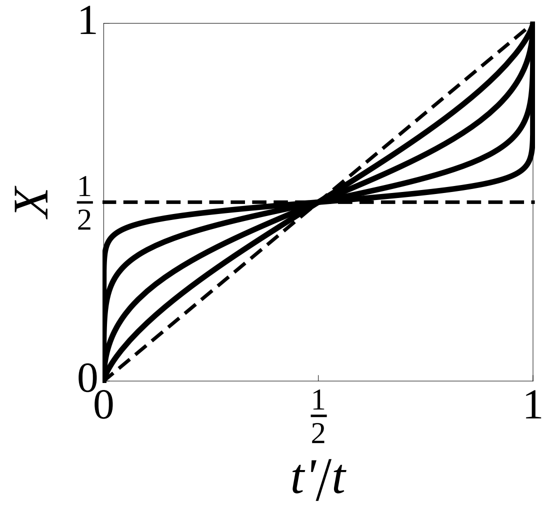


Figure 8. Fluctuation-dissipation ratio for the global order parameter (corresponding to $\vec{p} = 0$) at criticality $r = 0$ within the Gaussian approximation, as a function of the ratio $0 \leq t'/t \leq 1$ for various values of α. The straight horizontal and diagonal dashed lines correspond to $\alpha = 1$ and $\alpha = 0$, respectively. The solid curves, instead, correspond to $\alpha = 0.25$, 0.5, 0.75, and 0.9 upon moving away from the diagonal line.

(Note, however, that beyond the Gaussian approximation such a temperature depends upon the observable used to define it [44].) In equilibrium one has $X^\infty = 1$ (which is just a reformulation of the FDT) and $\beta^\infty = \beta$. Instead, $X^\infty \neq 1$ is a signal of an asymptotic non-equilibrium dynamics and therefore we shall focus on this quantity for the dynamics we are presently interested in.

Within the Gaussian approximation discussed in Sec. 3.1 the fluctuation-dissipation ratio X can be easily calculated from the expressions in Eqs. (3.7) and (3.8) [see also Eqs. (B.17) and (B.21)] for the response and correlation function, respectively, in terms of Mittag-Leffler functions:

$$X^{-1}(t,t') = 1 + \left(\frac{t}{t'} - 1\right)^{1-\alpha} \frac{E_\alpha(-At^\alpha/\gamma)E_{\alpha,\alpha}(-At'^\alpha/\gamma)}{E_{\alpha,\alpha}(-A(t-t')^\alpha/\gamma)}, \tag{4.43}$$

where we assumed $t > t'$ and $A \equiv p^2 + r$. For $A \neq 0$ (e.g., far from the critical point $r = 0$ or at criticality with $\vec{p} \neq 0$) and long and well-separated times $t, t', t - t' \gg (\gamma/A)^{1/\alpha}$, one can easily see from Eq. (B.19) that $X^{-1} \to 1$, confirming the expectation that the system equilibrates at long times, independently of the value of $\alpha > 0$. On the other hand, for the fluctuation of the homogeneous mode $\vec{p} = 0$ at criticality one has $A = 0$ and the FDR takes the simple form (originally derived in Ref. [28] for an anomalously diffusing particle)

$$X^{-1}_{\vec{p}=0,\mathrm{crit}}(t,t') = 1 + \left(\frac{t}{t'} - 1\right)^{1-\alpha} \tag{4.44}$$

which is a universal scaling function of the dimensionless scaling variable t'/t, reported in Fig. 8 for various values of α. In contrast to the white noise case $\alpha = 1$, in the

presence of colored noise $0 < \alpha < 1$, $X^{-1}_{\vec{p}=0,\mathrm{crit}}(t,t')$ does actually depend on t'/t and it interpolates continuously between the quasi-equilibrium regime $t' \simeq t$, within which $X_{\vec{p}=0,\mathrm{crit}} \simeq 1$, and the non-equilibrium regime of well separated times $t' \ll t$, for which $X_{\vec{p}=0,\mathrm{crit}} \simeq 0$ as it is generically observed in the case of coarsening dynamics [33, 34, 45].

Beyond the Gaussian approximation, we can deduce an expression of the two-time dependent FDR and its limiting values from the scaling forms in Eq. (4.22) and Eq. (4.23). First of all we note that for $t \gg t'$ and $\vec{p} = 0$ one has $\partial_{t'} C \simeq t^{(2-\eta)/z+\theta-\hat{\alpha}}(\hat{\alpha}-\theta)t'^{\hat{\alpha}-\theta-1}F_C(\vec{0},0)$ and $R \simeq t^{(2-\eta-z)/z}(t/t')^{\theta}F_R(\vec{0},0)$. We thus obtain

$$X^{\infty} = \frac{F_R(\vec{0},0)}{(\hat{\alpha}-\theta)F_C(\vec{0},0)} \lim_{t'\to\infty}\lim_{t\to\infty}\left(\frac{t}{t'}\right)^{\hat{\alpha}-1}. \tag{4.45}$$

In the case $\hat{\alpha} = 1$ of dominant white noise this expression renders the well-known result $X^{\infty} = F_R(\vec{0},0)/[(1-\theta)F_C(\vec{0},0)]|_{\alpha=1} \equiv X^{\infty}_{\mathrm{w}}$ [32, 14], i.e., $X^{\infty}_{\mathrm{w}} = 1/2$ within the Gaussian approximation [46, 47, 48]. The contribution of non-Gaussian fluctuations for $D < 4$ and up to the second order in the ϵ-expansion have been calculated in [36], in rather good agreement with Monte Carlo simulation (see Ref. [14] for a summary). Instead, if the colored noise dominates $\hat{\alpha} = \alpha < 1$ and therefore the long-time limit X^{∞} of the FDR in Eq. (4.45) vanishes, formally corresponding to an infinite effective temperature as observed in coarsening processes. Note that this result holds at all orders in perturbation theory. Therefore,

$$X^{\infty} = \begin{cases} X^{\infty}_{\mathrm{w}} & \text{for } \alpha > \alpha_c(D,N), \\ 0 & \text{for } \alpha < \alpha_c(D,N), \end{cases} \tag{4.46}$$

where both values do not depend on the actual value of α and therefore X^{∞} exhibits a discontinuity as a function of α upon crossing the line $\alpha = \alpha_c(D,N)$. Within the Gaussian approximation one can easily check the general result Eq. (4.46) for X^{∞}, on the basis of Eqs. (4.2) and (4.3). Indeed the behavior of the correlation and response functions can be determined by taking $\lim_{\lambda\to 0}\lambda C_0(\vec{0};\lambda,\kappa)$ and $\lim_{\lambda\to 0}\lambda R_0(\vec{0};\lambda,\kappa)$, respectively, for the propagators at zero momentum and at criticality. It is then straightfoward to obtain $X^{\infty}_0 = 1/2$ within region W and $X^{\infty}_0 = \lim_{\lambda\to 0}\Gamma_{\kappa}/\Gamma_{\lambda} = 0$ within region C, which confirms our general results. Apparently, this result for X^{∞}_0 contradicts the corresponding one $X^{\infty}_0 = 1$ for a freely diffusing particle in a super-Ohmic bath (corresponding to $\alpha > 1$) found in [28], which our model reduces to within the Gaussian approximation. However, within the field-theoretical approach discussed here, it turns out that a super-Ohmic bath, responsible for a noise Γ with $\alpha > 1$ in Eq. (2.6), is eventually controlled by the white-noise vertex and it is therefore unstable with respect to the effects of the interaction, which effectively generates such a vertex even though it was not present in the original coupling to the bath. Therefore, the white-noise result $X^{\infty}_0 = 1/2$ does not only apply to the cross-over line $\alpha_c(D,N)$ but it is valid within the whole region W. On the same footing, the results discussed here suggest that, at least in higher spatial dimensions, adding interactions to a system which displays superdiffusion (corresponding to $z < 2$) results quite generically in a

sub-diffusive behavior ($z > 2$) as expected in the case of a diffusing particle (Gaussian approximation) with interactions.

5. Summary and outlook

We studied the purely dissipative critical dynamics of a model with an N-component order parameter in D spatial dimensions, coupled to an *equilibrium* thermal bath which provides a colored thermal noise. We argued that the upper critical dimensionality of the model is $D_c = 4$ and we used the framework of the field-theoretical ϵ-expansion to account for the effects of non-Gaussian fluctuations in $4 - \epsilon$ spatial dimensions.

Within the Gaussian approximation — valid for $D > D_c$ — the equilibrium dynamic exponent z which controls the different scaling of space and time takes the values

$$z_0 = \begin{cases} z_0^{(\mathrm{col})} = 2/\alpha & \text{for } \alpha < 1, \\ z_0^{(\mathrm{w})} = 2 & \text{for } \alpha \geq 1, \end{cases} \tag{5.1}$$

where α characterises the *algebraic* long-time decay of the two-time correlation function of the noise, see Eq. (2.6). For $\alpha = 1$ one recovers the white-noise result $z_0^{(\mathrm{w})} = 2$. The non-equilibrium 'initial slip exponent' θ, instead, vanishes. Depending upon the value of α the asymptotic long-time dynamics is effectively equivalent to one driven by white noise (Ohmic bath) for $\alpha > \alpha_c$, wheras the effect of the colored noise is relevant for $\alpha < \alpha_c$. Within the Gaussian approximation $\alpha_c = 1$, as demonstrated by the change in behavior of z_0 given in Eq. (5.1).

In dimensions $D < 4$ the critical behavior is modified due to the relevance of the interaction term and of the non-Gaussian fluctuations. The value α_c which controls the cross-over between the white-noise and the colored-noise dominated behaviors is modified by N-dependent corrections of order ϵ^2 and it therefore separates the two corresponding regions in the parameter space (α, D, N), named W and C in Fig. 4, respectively. The dynamical critical exponent z is given by

$$z = \begin{cases} z^{(\mathrm{col})} \equiv \dfrac{2}{\alpha} + \eta_\gamma = \dfrac{2}{\alpha}\left[1 - \dfrac{N+2}{4(N+8)^2}\epsilon^2\right] + \mathcal{O}(\epsilon^3) & \text{within region C,} \\ z^{(\mathrm{w})} \equiv 2 + \eta_{\mathrm{w}} = 2 + \dfrac{N+2}{(N+8)^2}\left[3\ln\dfrac{4}{3} - \dfrac{1}{2}\right]\epsilon^2 + \mathcal{O}(\epsilon^3) & \text{within region W.} \end{cases} \tag{5.2}$$

The N-dependent curve (3.32) which separates regions W and C in the (α, D)-plane is illustrated in Fig. 4 for $N = 1, 4, \infty$. Some comments are in order:

(i) Upon decreasing D, the region W within which the Ohmic result is recovered extends beyond the Gaussian value $\alpha_c = 1$.

(ii) The correction to the Gaussian value z_0 is positive within region W ($z_0 = 2$) and negative within region C ($z_0 = 2/\alpha$).

(iii) The exponent z is a continuous function of ϵ and α: At the transition line between regions W and C one has $z^{(\mathrm{w})} = z^{(\mathrm{col})}$, as can be easily verified by using Eq. (3.32).

(iv) In the large-N limit the ϵ^2 correction vanishes and the dynamic exponent z and α_c take their Gaussian values z_0 and $\alpha_c = 1$, respectively.

For random initial conditions, i.e., with vanishing correlations and average order parameter, we determined the general scaling forms of the dynamic correlation functions. Within region C, such scaling forms differ from the ones valid in the presence of white noise only, studied in Ref. [10] and recovered within region W. We determined the corresponding initial-slip exponent θ up to order $\mathcal{O}(\epsilon)$ in the presence of colored noise. It is given by

$$\theta = \frac{(N+2)}{4(N+8)} d(\alpha)\Gamma_E(\alpha)\epsilon + \mathcal{O}(\epsilon^2), \tag{5.3}$$

and the plot of the ratio between this value θ and the reference $\theta_{\alpha=1}$ for the white noise is reported in Fig. 7.

In non-equilibrium conditions we also calculated the long-time limit X^∞ of the FDR for general α and N. The value of X^∞ in the presence of white noise is known analytically up to $\mathcal{O}(\epsilon^2)$ [36] and numerically via Monte Carlo simulations in various dimensions for models belonging to the universality class of the $O(N)$ model with dissipative dynamics (see, e.g., [14] for a review). We proved that this result is recovered within region W. Instead, if the colored noise is dominant $[\alpha < \alpha_c(D,N)]$, i.e., within region C, we showed that $X^\infty = 0$. Therefore, the associated effective temperature is infinite, analogously to what is found in sub-critical coarsening [33, 34]. Our result for X^∞ within the Gaussian approximation is only in partial agreement with the corresponding one derived in [28] for an anomalously diffusing particle — i.e., of a fractional Brownian motion — which our model reduces to within such an approximation. Indeed, in the presence of a super-Ohmic noise $\alpha > \alpha_c = 1$, one finds $X^\infty = 1$ [28] and super-diffusion $z < 2$ for the fractional Brownian motion, while we argue that $X^\infty = X_0^\infty = 1/2$ and normal diffusion $z = z_0^{(\mathrm{w})} = 2$ in our field theoretical model. This is due to the fact that even in the absence of a white-noise effective vertex in the original model, non-Gaussian fluctuations (induced by the interactions) generate it and turn it into the dominant one for $\alpha > \alpha_c \leq 1$ such that the white-noise result is recovered.

In conclusion, noises correlated in time may affect significantly the equilibrium and non-equilibrium dynamical properties of systems close to critical points. In this respect it is important to note that the distinction between super-Ohmic ($\alpha > 1$) and sub-Ohmic ($\alpha < 1$) thermal baths does not fully correspond to having irrelevant (white) and relevant (colored) long-time correlations of the noise, respectively. Indeed, as shown in Fig. 4, even a weakly sub-Ohmic noise with $\alpha_c(D,N) < \alpha < 1$ is actually equivalent (in the RG sense) to an Ohmic (white) noise in the physical dimensions $D = 3$ and $D = 2$ as far as the dynamical properties in the long-time limit are concerned. In addition, in the presence of interactions, a super-Ohmic bath does not result in a super-diffusive behavior ($z < 2$) but rather in the anomalous diffusion induced by the equivalent white noise, in contrast to what happens for the free fractional Brownian motion.

The field-theoretical predictions for the relaxational Markov critical dynamics of systems belonging to the universality class considered here have been put to the

numerical test both via Monte Carlo simulations and by solving the Langevin equations with a variety of different methods (see, e.g., [15] and references therein). An instance of non-Markovian dynamics of the ϕ^4-theory with a noise exponentially correlated in time was investigated in [49]. However, in this case one does not expect the long-time dynamics of the system to be affected by the finite memory of the noise. Dealing numerically with power-law correlated Gaussian noise is a significantly harder problem which remains basically open due to the difficulties in generating such kind of random process, see, e.g., [27, 50] and references therein.

One of the virtues of the approach we have followed here is that it can be easily applied to quantum critical dynamics [51]. For instance, the thermal bath can be modeled by a set of (quantum) harmonic oscillators coupled to all degrees of freedom of the system. Within the Schwinger-Keldysh formalism it is possible to derive a path-integral representation of the non-equilibrium dynamics [18, 52]. Integrating out the oscillator variables one obtains an action similar to the one considered here [18, 52, 53]. The main difference with the classical case is that even Ohmic dissipation leads to retarded interactions. The present work is expected to provide at least a partial and preliminary insight into the more difficult problem of the analysis of quantum critical equilibrium and non-equilibrium dissipative dynamics [51].

Among other possible extensions of the present work, we mention the problem of understanding the effects of colored noise on sub-critical coarsening. The dynamic scaling hypothesis states that the late-stage phase ordering kinetics is governed by a length scale $L(t)$ that, in models with no quenched disorder, typically grows in time as a power-law $L(t) \simeq \lambda(T)t^{1/z_d}$. The dynamic exponent z_d (generically different for the dynamic exponent z at criticality) depends upon the kind of order parameter and the conservation laws [54] while the prefactor $\lambda(T)$ typically depends only weakly upon temperature T, is non-universal, and it vanishes upon approaching a critical point. (The matching with the critical growth is explained in [55].) In presence of colored noise this growth law might be modified, even though one usually expects thermal fluctuations not to affect the domain growth [54]. We shall address this issue in a future study.

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A. Fourier and Laplace conventions

Within the present study we define the Fourier transform and its inverse via

$$\hat{F}(\omega) = \int_{-\infty}^{\infty} \mathrm{d}t\ \mathrm{e}^{-i\omega t}\ F(t), \qquad \text{and} \qquad F(t) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi}\ \mathrm{e}^{i\omega t}\ \hat{F}(\omega). \quad \text{(A.1)}$$

Instead, for every $\lambda > 0$ the Laplace transform is defined as

$$\hat{F}_\lambda = \int_0^{\infty} \mathrm{d}t\ \mathrm{e}^{-\lambda t}\ F(t). \quad \text{(A.2)}$$

In the main text we shall drop the hats, whenever this does not generate confusion.

B. The equilibrium propagators

B.1. Scaling in real time

For $\alpha = 1$ (white noise) the equilibrium propagators have a simple analytic form in the time domain [3, 10, 35]. They can be calculated by applying an inverse Fourier transform to Eqs. (3.4) and (3.5):

$$R_0(\vec{p}, t) = \Theta(t)\mathrm{e}^{-(p^2+r)t/\gamma_\mathrm{w}} \quad \text{(B.1)}$$

$$C_0(\vec{p}, t) = \frac{1}{p^2 + r}\mathrm{e}^{-(p^2+r)|t|/\gamma_\mathrm{w}}. \quad \text{(B.2)}$$

For general α dimensional analysis suggests that the critical ($r = 0$) Gaussian correlation C_0 with $\gamma_\mathrm{w} = 0$ should scale as

$$C_0(\vec{p}, t) = p^{-2}\ f_{C_0}(p^2|t|^\alpha/\gamma). \quad \text{(B.3)}$$

Using Eqs. (3.3) and (3.5) the equal-time correlator is given by

$$\begin{aligned} C_0(\vec{p}, t = 0) &= \int \frac{\mathrm{d}\omega}{2\pi}\ C_0(\vec{p}, \omega) \\ &= \int \frac{\mathrm{d}\omega}{2\pi}\ \frac{2\gamma \sin(\pi\alpha/2)|\omega|^{\alpha-1}}{\gamma^2|\omega|^{2\alpha} + 2\gamma(p^2 + r)|\omega|^\alpha \cos(\pi\alpha/2) + (p^2 + r)^2} \\ &= \frac{1}{p^2 + r}. \end{aligned} \quad \text{(B.4)}$$

Hence, we infer that $f_{C_0}(0) = 1$. Naturally, we have $f_{C_0}(\infty) = 0$ since correlations have to vanish in the long-time limit. Applying a Fourier transform to Eq. (B.3) it is easy to show that at criticality ($r = 0$)

$$C_0(\vec{x}, t) = \frac{1}{|x|^{D-2}}\ g_{C_0}(\gamma x^2/|t|^\alpha), \quad \text{(B.5)}$$

where the function g_{C_0} reaches the asymptotic value $g_{C_0}(\infty) = \Gamma_E(D/2-1)/(4\pi^{D/2})$. In order to deduce the leading behavior for $g_{C_0}(u)$ when $u \to 0$ we start from the explicit expression of the noise kernel $\Gamma_{i\omega}$ given in Eq. (3.3). After some algebra we obtain

$$g_{C_0}(u) = \int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{\mathrm{d}\omega}{2\pi}\ \frac{2u \sin(\pi\alpha/2)|\omega|^{\alpha-1}\mathrm{e}^{i\omega + i\vec{p}\cdot\hat{z}}}{u^2|\omega|^{2\alpha} + 2up^2|\omega|^\alpha \cos(\pi\alpha/2) + p^4}, \quad \text{(B.6)}$$

where $u = \gamma x^2/t^\alpha$ and $\hat{z}$ is an arbitrary unit vector. For $\alpha \leq 1$ we neglect the contributions of $\mathcal{O}(u^2)$ in the denominator and we obtain

$$g_{C_0}(u \to 0) = 2u \int \frac{\mathrm{d}\omega}{2\pi}\, |\omega|^{\alpha-1} \mathrm{e}^{i\omega} \int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{\sin(\pi\alpha/2)\mathrm{e}^{i\vec{p}\cdot\hat{z}}}{(p^2 + u|\omega|^\alpha \cos(\pi\alpha/2))^2}. \quad \text{(B.7)}$$

The integral over $\vec{p}$ is of $\mathcal{O}(u \ln[u|\omega|^\alpha \cos\pi\alpha/2])$ for $D = 4$ and the resulting integral converges for $\alpha < 1$; consequently,

$$g_{C_0}(u \to 0) \sim \mathcal{O}(u \ln u). \quad \text{(B.8)}$$

By using FDT we derive

$$R_0(\vec{x}, t) = \frac{\alpha\gamma}{x^{D-4} t^{\alpha+1}}\, g'_{C_0}(\gamma x^2/t^\alpha)\Theta(t). \quad \text{(B.9)}$$

In the white-noise case, the scaling function g_{C_0} has the simple form

$$g_{C_0}(u) = \frac{\Gamma_E(D/2-1)}{4\pi^{D/2}} \left[1 - \frac{\Gamma_E\left(\frac{D}{2}-1, \frac{u}{4}\right)}{\Gamma_E\left(\frac{D}{2}-1\right)} + \mathcal{O}(\epsilon)\right], \quad \text{(B.10)}$$

with $\Gamma_E(s, x) = \int_x^\infty \mathrm{d}y\; y^{s-1}\mathrm{e}^{-t}$, whence we deduce for $\alpha = 1$ and $D = 4$

$$g_{C_0}(u \to 0) = \mathcal{O}(u). \quad \text{(B.11)}$$

For generic γ and γ_{w} the scaling function g_{C_0} is no longer a function of one variable. It is easy to show that

$$C_0(\vec{x}, t) = \frac{1}{|x|^{D-2}}\, g_{C_0}(\gamma|x|^2/|t|^\alpha, \gamma_{\mathrm{w}}|x|^2/|t|). \quad \text{(B.12)}$$

Moreover, by using a similar argument as above one has for $D = 4$

$$\lim_{t\to\infty} g_{C_0}(u, v) = \mathcal{O}((u + v)\ln u), \quad \text{(B.13)}$$

where $u = \gamma|x|^2/t^\alpha$ and $v = \gamma_{\mathrm{w}}|x|^2/t$ vanish with u/v finite. In the opposite short-time limit in which u and v diverge with u/v finite,

$$\lim_{t\to 0} g_{C_0}(u, v) = \Gamma_E(D/2-1)/(4\pi^{D/2}) \quad \text{(B.14)}$$

as for the purely colored problem.

The equilibrium propagators can be written in terms of the generalized Mittag-Leffler functions $E_{a,b}(z)$, as discussed in App. B.2.

B.2. Generalized Mittag-Leffler functions

The Laplace transform of $R_0(\vec{p}, t)$ is given by

$$R_0(\vec{p}, \lambda) = \frac{1}{\lambda\Gamma_\lambda + A} \quad \text{(B.15)}$$

where we defined $A \equiv p^2 + r$ and, in the case of colored noise, $\Gamma_\lambda = \gamma\lambda^{\alpha-1}$. We formally expand this expression for small A:

$$R_0(\vec{p}, \lambda) = \frac{1}{\gamma\lambda^\alpha} \frac{1}{1 + A(\gamma\lambda^\alpha)^{-1}} = \frac{1}{\gamma\lambda^\alpha} \sum_{k=0}^{\infty} \frac{(-A/\gamma)^k}{\lambda^{\alpha k}} \quad \text{(B.16)}$$

where the terms of the form $1/\lambda^\beta$ (with $\mathrm{Re}\,\beta > 0$) are recognized as the Laplace transform of $\Theta(t)t^{\beta-1}/\Gamma_E(\beta)$, so that (B.16) is identified as the Laplace transform of

$$\begin{aligned} R_0(\vec{p},t) &= \Theta(t)\frac{1}{\gamma}\sum_{k=0}^{\infty}(-A/\gamma)^k\frac{t^{\alpha k+\alpha-1}}{\Gamma_E(\alpha k+\alpha)} \\ &= \Theta(t)\frac{t^{\alpha-1}}{\gamma}E_{\alpha,\alpha}(-At^\alpha/\gamma), \end{aligned} \tag{B.17}$$

where we have introduced the generalized Mittag-Leffler function

$$E_{\alpha,\beta}(z) \equiv \sum_{k=0}^{\infty}\frac{z^k}{\Gamma_E(\alpha k+\beta)} \quad \text{with} \quad \alpha,\beta,z\in\mathbb{C},\ \mathrm{Re}\{\alpha,\beta\}>0. \tag{B.18}$$

Note that this function reduces to an exponential for $\alpha=\beta=1$: $E_{1,1}(z)=\mathrm{e}^z$, whereas for $z\in\mathbb{R}$ [43],

$$E_{\alpha,\beta}(z\to-\infty) = -\sum_{k=1}^{k^*}\frac{1}{\Gamma_E(\beta-\alpha k)}\frac{1}{z^k} + \mathcal{O}(z^{-(k^*+1)}). \tag{B.19}$$

The corresponding expression for the equilibrium Gaussian correlation function can be obtained from the FDT (2.12). Indeed, after integration Eq. (2.12) takes the form

$$C_0(\vec{p},t) = C_0(\vec{p},t=0) - \int_0^{|t|}\mathrm{d}s\, R_0(\vec{p},s) \tag{B.20}$$

where we used the fact that, in equilibrium, $C(\vec{x},t)=C(\vec{x},-t)$. Taking into account (B.4) and the first line of (B.17) one readily finds

$$C_0(\vec{p},t) = \frac{1}{A}E_\alpha(-A|t|^\alpha/\gamma) \tag{B.21}$$

where $E_\alpha(z)\equiv E_{\alpha,1}(z)$ is the Mittag-Leffler function.

The correlation function $C_0(\vec{p},t)$ in Eq. (B.21) can also be expressed as the inverse Fourier transform of $C_0(\vec{p},\omega)$ reported in Eq. (3.5) (see also Eq. (3.3)). After some suitable change of variables one finds the following scaling form

$$C_0(\vec{p},t) = \frac{1}{A}f_{C_0}(A|t|^\alpha/\gamma), \tag{B.22}$$

where

$$\begin{aligned} f_{C_0}(u) &\equiv \frac{2}{\pi}\int_0^\infty\mathrm{d}v\,\cos(u^{1/\alpha}v)\frac{v^{\alpha-1}\sin(\pi\alpha/2)}{v^{2\alpha}+2v^\alpha\cos(\pi\alpha/2)+1} \\ &= \frac{\sin(\pi\alpha/2)}{\pi\alpha/2}\int_0^\infty\mathrm{d}v\frac{\cos(u^{1/\alpha}v^{1/\alpha})}{v^2+2v\cos(\pi\alpha/2)+1} \end{aligned} \tag{B.23}$$

is the explicit expression for the scaling function introduced in Eq. (B.3).

C. Calculation of $\mathcal{E}^{0,2}_{\mathrm{w}}$ and $\mathcal{E}^{1,1}$

Starting from Eq. (3.21) we have for generic γ and γ_{w}

$$u^2\mathcal{E}^{0,2}(\sigma;\gamma,\gamma_{\mathrm{w}}) = \ell\ \frac{g^2A_D(N+2)}{9}$$

$$\times \left\{ z\ell^{z-1}\cos(\sigma\ell^z) \int_\ell^\infty \mathrm{d}x\, x^{5-2D}\; g_{C_0}^3\left(\frac{\gamma x^2}{\ell^{\alpha z}}, \frac{\gamma_\mathrm{w} x^2}{\ell^z}\right) + \int_{\ell^z}^\infty \mathrm{d}t\, \cos(\sigma t)\, \ell^{5-2D}\; g_{C_0}^3\left(\frac{\gamma \ell^2}{t^\alpha}, \frac{\gamma_\mathrm{w}\ell^2}{t}\right)\right\}. \tag{C.1}$$

The result of the integral in the first term in curly brackets is an analytic function of σ that admits a Taylor expansion in powers of σ^2, i.e.,

$$c_0 + c_2\sigma^2 + c_4\sigma^4 + \ldots \tag{C.2}$$

with coefficients that, in principle, depend separately on γ, γ_w and ℓ. The integral in the second term in curly brackets yields, instead, a non-analytic function of σ that we can still express as a series:

$$d_0 + d_2\sigma^2 + \ldots + d_{3\alpha-1}\sigma^{3\alpha-1} + \ldots \tag{C.3}$$

where the term $\propto \sigma^{3\alpha-1}$ is due to the leading behavior of $g_{C_0}^3$ for $t \to +\infty$ [see Eq. (B.8)] which has to be subtracted for $\alpha < 1/3$ in order to make the integral convergent at large t. If $3\alpha - 1 > 0$ the limit $\sigma \to 0$ can be safely taken and the white-noise vertex is renormalized by $c_0 + d_0$. If, on the contrary, $3\alpha - 1 < 0$ the contribution proportional to $\sigma^{3\alpha-1}$ is anyhow negligible (for $\alpha > 0$) with respect to the term $\gamma\sigma^{\alpha-1}$ which is already present in the tree-level vertex. Therefore, there is no renormalization of the colored-noise vertex and we can focus on the limit $\gamma_\mathrm{w} \gg \gamma$, i.e., on the correction to the white-noise vertex only. Since we calculate evolution equations up to order ϵ^2 we simply need to evaluate (C.1) in $D = 4$. We obtain

$$\begin{aligned} u^2\mathcal{E}^{0,2}(0;\gamma,\gamma_\mathrm{w}) = \ell\, \frac{g^2 A_D(N+2)}{9} & \times \left\{ z\ell^{z-1} \int_\ell^\infty \mathrm{d}x\, x^{-3}\; g_{C_0}^3\left(\frac{\gamma x^2}{\ell^{\alpha z}}, \frac{\gamma_\mathrm{w} x^2}{\ell^z}\right) \right. \\ & \left. + \int_{\ell^z}^\infty \mathrm{d}t\, \ell^{-3}\; g_{C_0}^3\left(\frac{\gamma \ell^2}{t^\alpha}, \frac{\gamma_\mathrm{w}\ell^2}{t}\right)\right\}. \end{aligned} \tag{C.4}$$

We are interested in the $\alpha \to \alpha_c$ limit in which $\gamma_\mathrm{w} \to \infty$ and $z = 2 + \mathcal{O}(\epsilon^2)$. By first using $x \mapsto x\ell/\sqrt{\gamma_\mathrm{w}}$ and $t \mapsto \gamma_\mathrm{w}\ell^2/x^2$ we transform the two-variable scaling function into the one-variable white-noise one. Using then Eq. (B.10) and $A_4 = 2\pi^2$ we obtain the second and third line below.

$$\begin{aligned} u^2\mathcal{E}^{0,2}(0;\gamma,\gamma_\mathrm{w}) &= \frac{2\gamma_\mathrm{w} g^2 A_D(N+2)}{9}\left[\int_{\sqrt{\gamma_\mathrm{w}}}^\infty \mathrm{d}x\, x^{-3} g_{C_0}^3(0,x^2) + \int_0^{\sqrt{\gamma_\mathrm{w}}} \mathrm{d}x\, x^{-3} g_{C_0}^3(0,x^2)\right] \\ &= \frac{2\gamma_\mathrm{w} u^2(N+2)}{9}\int_0^\infty \mathrm{d}x\left[1 - \mathrm{e}^{-x^2/4}\right]^3/x^3 \\ &= \frac{\gamma_\mathrm{w} u^2(N+2)}{12}\ln\frac{4}{3}. \end{aligned} \tag{C.5}$$

Therefore, at the critical point, using the Wilson-Fisher fixed point value $u^* = 6\epsilon/(N+8) + \mathcal{O}(\epsilon^2)$ [37], we find

$$u^2\mathcal{E}^{0,2}(0;\gamma,\gamma_{\rm w}) \to u^{*2}\gamma_{\rm w}\mathcal{E}^{0,2}_{\rm w} = \gamma_{\rm w}\,\frac{3(N+2)}{(N+8)^2}\,\ln\frac{4}{3}\,\epsilon^2 + \mathcal{O}(\epsilon^3). \tag{C.6}$$

We now compute $\mathcal{E}^{1,1}$ in $D = 4$. We start from Eqs. (3.39) and (3.40). Using Eqs. (D.5) and (B.14) in the limit $\ell \to 0$ we obtain

$$\begin{aligned} u^2\mathcal{E}^{1,1}(0;\gamma,\gamma_{\rm w}) &= \frac{g^2 A_4(N+2)\pi}{144}\frac{-\partial}{\partial\ln\ell}\int_\ell^\infty \frac{\mathrm{d}x}{x}\frac{1}{(2\pi)^6} \\ &= \frac{u^2(N+2)}{72}. \end{aligned} \tag{C.7}$$

Note that the term coming from the differentiation of $C_0(\vec{x},\ell^z)$ in Eq. (3.39) with respect to $\ln\ell$ vanishes in the limit $\ell \to 0$ [use Eq. (B.5)]. At the critical point we obtain

$$u^{*2}\mathcal{E}^{1,1} = \frac{u^{*2}(N+2)}{72} = \frac{N+2}{2(N+8)^2}\,\epsilon^2 + \mathcal{O}(\epsilon^3). \tag{C.8}$$

D. Non-equilibrium propagators

For $\alpha = 1$ the Gaussian non-equilibrium propagators read in the momentum and time domain

$$C_0(\vec{p};t,s) = \frac{1}{p^2+r}\left[\mathrm{e}^{-(p^2+r)|t-s|/\gamma_{\rm w}} - \mathrm{e}^{-(p^2+r)(t+s)/\gamma_{\rm w}}\right], \tag{D.1}$$

$$R_0(\vec{p};t,s) = \Theta(t-s)\mathrm{e}^{-(p^2+r)(t-s)/\gamma_{\rm w}}. \tag{D.2}$$

For generic α, however, analogously compact expressions are not available and our analysis proceeds using Laplace transforms. In order to determine the response function R_0 — consistently with the Gaussian approximation — we start with the linearized version of the Langevin equation Eq. (2.2) in the presence of an external perturbation $\vec{h}$:

$$\int_0^t \mathrm{d}t'\;\Gamma(t-t')\partial_{t'}\vec{\phi}(\vec{x},t') + (r-\nabla^2)\vec{\phi}(\vec{x},t') = \vec{\zeta}(\vec{x},t) + \vec{h}(\vec{x},t) \tag{D.3}$$

Calculating the expectation value of both sides with respect to the distribution of the noise eliminates the vanishing average $\langle\vec{\zeta}\rangle$. The Laplace transform yields

$$(\lambda\Gamma_\lambda + p^2 + r)\langle\vec{\phi}_\lambda(\vec{p})\rangle_h = \vec{h}_\lambda(\vec{p}) \tag{D.4}$$

in momentum space where we used the Dirichlet boundary condition $\phi(\vec{x},t=0) = 0$ [see discussion at the beginning of Sec. 4.1]. Note that the expectation value of the order parameter depends on h. The response propagator in the Laplace domain is given by

$$\begin{aligned} R_0(\vec{p};\lambda,\kappa)\delta_{ij} &= \frac{\delta\langle\phi_{i,\lambda}(\vec{p})\rangle_h}{\delta h_{j,\kappa}}\Big|_{\vec{h}=\vec{0}} = \frac{1}{\lambda\Gamma_\lambda + p^2 + r}\frac{\delta h_{i,\lambda}}{\delta h_{j,\kappa}} \\ &= \frac{1}{(\lambda+\kappa)(\lambda\Gamma_\lambda + p^2 + r)}\delta_{ij}. \end{aligned} \tag{D.5}$$

The last equality follows from the fact that $\delta h_i(t)/\delta h_j(s) = \delta_{ij}\delta(t-s)$ as a function of time translates into $\delta_{ij}/(\lambda+\kappa)$ in Laplace space, given that $\int_0^\infty \mathrm{d}t\mathrm{d}s\, \mathrm{e}^{-\lambda t-\kappa s}\delta(t-s) = 1/(\lambda+\kappa)$.

In order to deduce the correlation propagator we start directly from Eq. (D.3) with $\vec{h}=0$ and we consider its Laplace transform:

$$\vec{\phi}_\lambda(\vec{p}) = \frac{\vec{\zeta}_\lambda}{\lambda\Gamma_\lambda + p^2 + r} \tag{D.6}$$

which yields

$$\begin{aligned} C_0(\vec{p};\lambda,\kappa)\delta_{ij} = \langle \phi_{i,\lambda}(\vec{p})\phi_{j,\kappa}(-\vec{p})\rangle &= \frac{\langle \zeta_{i,\lambda}\zeta_{j,\kappa}\rangle}{(\lambda\Gamma_\lambda + p^2 + r)(\kappa\Gamma_\kappa + p^2 + r)} \\ &= \frac{\Gamma_\lambda + \Gamma_\kappa}{(\lambda+\kappa)(\lambda\Gamma_\lambda + p^2 + r)(\kappa\Gamma_\kappa + p^2 + r)}\delta_{ij}. \end{aligned} \tag{D.7}$$

In the last line we used the fact that $\int_0^\infty \mathrm{d}t\mathrm{d}s\, \mathrm{e}^{-\lambda t-\kappa s}\Gamma(t-s) = (\Gamma_\lambda+\Gamma_\kappa)/(\lambda+\kappa)$. The propagators verify an 'initial time FDT'. We see that for $\alpha<1$ $\lim_{\kappa\to\infty} \kappa R_0(\vec{p};\lambda,\kappa) = R_0(\vec{p},\lambda)$ and $\lim_{\kappa\to\infty} \kappa^2 C_0(\vec{p};\lambda,\kappa) = \lim_{k\to\infty} \kappa^{1-\alpha}\Gamma_\lambda R_0(\vec{p},\lambda)$, with $R_0(\vec{p},\lambda) = 1/(\lambda\Gamma_\lambda + p^2 + r)$. In the time domain, the second identity reads

$$\partial_{t'} C_0(\vec{p};t,t'\to 0) \sim t'^{\alpha-1}\int_0^t \mathrm{d}s\,\Gamma(t-s)R_0(\vec{p};s,t'\to 0). \tag{D.8}$$

To derive this equation we used the convolution theorem for the Laplace transform $\mathcal{L}$, that is $\mathcal{L}\left[\int_0^t \mathrm{d}t'\ f(t-t')g(t')\right](\lambda) = \mathcal{L}[f](\lambda)\mathcal{L}[g](\lambda)$. In order to deduce the scaling of Eq. (D.8) with respect to t' one observes that if $\lambda\mathcal{L}[f(t)](\lambda) \sim \lambda^a$ for $\lambda\to\infty$ then $f(t)\sim t^{-a}$ for $t\to 0$.

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